Approximations for the $\sin_{[t]}(x)$ extrema

Notes by T Curtright, August 2011

It is amusing that the peak absolute value of the $\sin_{[t]}(x)$ iterates, obtained at $x = \pi/2 \mod \pi$, is well-approximated by the simple expression: 
$$\exp\left[(1 - \sqrt{t}) \ln (\pi/2)\right].$$
That is to say,
$$\sin_{[t]}(\pi/2) \approx \left(\frac{\pi}{2}\right)^{1-\sqrt{t}} = \frac{\pi}{2} \exp\left(-\sqrt{t} \ln (\pi/2)\right), \quad \ln (\pi/2) = 0.4515827053. \quad (10)$$

At least, this is true for $0 \leq t \leq 1$, where the relative error between the exact (numerical) value of $\sin_{[t]}(\pi/2)$ and this approximation is less than about 3 parts per mille. The branch point at $t = 0$ exhibited in this approximate expression is perhaps the simplest numerical evidence that the iterates are not analytic at $t = 0$ for all $x$.

\[
\begin{align*}
\sin_{[t]}(\pi/2) &\text{ compared to } \left(\frac{\pi}{2}\right)^{1-\sqrt{t}}, \text{ in blue, and } \left(\frac{\pi}{2}\right)^{1-\sqrt{t}} (1 + \text{poly}_7(\sqrt{t})), \text{ in green. Small circles are exact integer iterates of the sine.}
\end{align*}
\]

A more accurate fit for $0 \leq t \leq 1$ is given by
$$\sin_{[t]}(\pi/2) \approx \left(\frac{\pi}{2}\right)^{1-\sqrt{t}} \left(1 + 0.030989\sqrt{t} - 0.086989t + 0.045189(\sqrt{t})^3 + 0.049451t^2 - 0.061511(\sqrt{t})^5 + 0.027666t^3 - 0.004828(\sqrt{t})^7\right)$$

The relative error in this last expression is only a few parts per million (comparable to the uncertainty in the computation of the LHS using six conjugations of the series) as can be seen by evaluating the terms in parenthesis at $t = 1$ to obtain $0.999997$.\(^2\)

\(^2\) Actually, the coefficients fit to twelve digits give
$$1 + 0.030988758456\sqrt{t} - 0.086985976688t + 0.045188732627\sqrt{t}^3 + 0.049451109664t^2 - 0.061510544845\sqrt{t}^5 + 0.027695684952t^3 - 0.004827764218\sqrt{t}^7\bigg|_{t=1} = 1.00000000$$
$\sin_{[t]}(\pi/2)$ compared to $\left(\frac{\pi}{2}\right)^{1-\sqrt{t}}$, in blue, and $\left(\frac{\pi}{2}\right)^{1-\sqrt{t}} \left(1 + \text{poly}_7(\sqrt{t})\right)$, in green.
Relative error: \( 1 - \left( \frac{\pi}{2} \right)^{1 - \sqrt{t}} / \sin(|t| \pi/2) \)
Relative error: $1 - \left( \frac{\pi}{2} \right)^{1 - \sqrt{t}} \left( 1 + \text{poly}_7(\sqrt{t}) \right) / \sin(t/\pi)$, using twelve digit polynomial coefficients.