What is a derivative? Can you define it by defining and using the word "slope?" Yes, but how do you start from just the picture of the slope as the tangent of some angle and show that the derivative of x^2 with respect to x is $2x$? It can be done, but you've probably never seen such a derivation. Try it if you like.

The standard way that gets around this problem and that lets you apply it to many situations is

$$\frac{df}{dx}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

When you do a derivative, you will either recognize the answer because you have memorized it or you will use one of a few standard tools:

product rule, chain rule, linearity

and that's about it.

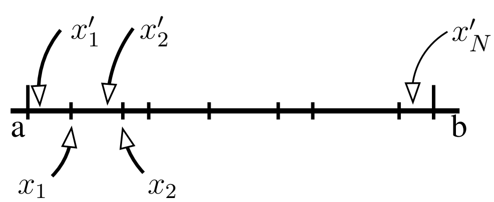
What is an integral? You've been using them for some time. I've been using the concept as if it's something that everyone knows. But what *is* it?

If your answer is something like "the function whose derivative is the given function" or "the area under a curve" then No. Both of these answers express an aspect of the subject but neither is a complete answer. The first actually refers to *the fundamental theorem of calculus,* and I'll describe that shortly. The second is a good picture that applies to some special cases, but it won't tell you how to compute it and it won't allow you to generalize the idea to the many other subjects in which it is needed. There are several different definitions of the integral, and every one of them requires more than a few lines to explain. I'll use the most common definition, the *Riemann Integral*.

A standard way to picture the definition is to try to find the area under a curve. You can get successively better and better approximations to the answer by dividing the area into smaller and smaller rectangles — ideally, taking the limit as the number of rectangles goes to infinity.

To codify this idea takes a sequence of steps:

1. Pick an integer $N > 0$. This is the number of subintervals into which the whole interval between a and b is to be divided.



2. Pick $N-1$ points between a and b. Call them x_1, x_2, etc.

$$a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$$

where for convenience I label the endpoints as x_0 and x_N. For the sketch , $N = 8$.

3. Let $\Delta x_k = x_k - x_{k-1}$. That is,

$$\Delta x_1 = x_1 - x_0, \qquad \Delta x_2 = x_2 - x_1, \cdots$$

4. In each of the N subintervals, pick one point at which the function will be evaluated. I'll label these points by the letter x'.

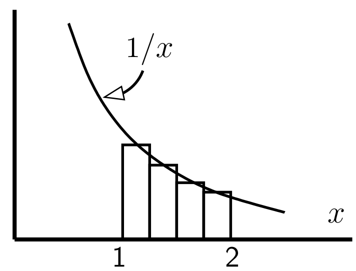
$$x_{k-1} \le x'_k \le x_k$$
$$x_0 \le x'_1 \le x_1, \qquad x_1 \le x'_2 \le x_2, \cdots$$

5. Form the sum that is an approximation to the final answer.

$$f(x_1')\Delta x_1 + f(x_2')\Delta x_2 + f(x_3')\Delta x_3 + \cdots$$

6. Finally, take the limit as all the $\Delta x_k \to 0$ and necessarily then, as $N \to \infty$. These six steps form the definition

$$\lim_{\Delta x_k \to 0} \sum_{k=1}^{N} f(x_k')\Delta x_k = \int_a^b f(x)\,dx$$



To demonstrate this numerically, pick a function and do the first five steps explicitly. Pick $f(x) = 1/x$ and integrate it from 1 to 2. The exact answer is the natural log of 2: $\ln 2 = 0.69315\ldots$

(1) Take $N = 4$ for the number of intervals

(2) Choose to divide the distance from 1 to 2 evenly, at $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$

$$a = x_0 = 1. < 1.25 < 1.5 < 1.75 < 2. = x_4 = b$$

(3) All the Δx's are equal to 0.25.

(4) Choose the midpoint of each subinterval. This is the best choice when you use only a finite number of divisions.

$$x_1' = 1.125 \qquad x_2' = 1.375 \qquad x_3' = 1.625 \qquad x_4' = 1.875$$

(5) The sum approximating the integral is then

$$f(x_1')\Delta x_1 \; + f(x_2')\Delta x_2 + f(x_3')\Delta x_3 + f(x_4')\Delta x_4 =$$
$$\frac{1}{1.125} \times .25 + \frac{1}{1.375} \times .25 + \frac{1}{1.625} \times .25 + \frac{1}{1.875} \times .25 = .69122$$

For such a small number of divisions, this is a very good approximation — about 0.3% error. (What do you get if you take $N = 1$ or $N = 2$ or $N = 10$ divisions?)

Fundamental Theorem of Calculus:

If the function that you're integrating is complicated or if the function is itself not known to perfect accuracy then the numerical approximation that I just did for $\int_1^2 dx/x$ is often the best way to go. How can a function not be known completely? If it's experimental data. When you have to resort to this arithmetic way to do integrals, are there more efficient ways to do it than simply using the definition of the integral? Yes. That's part of the subject of numerical analysis.

The fundamental theorem of calculus unites the subjects of differentiation and integration. The integral is defined as the limit of a sum. The derivative is defined as the limit of a quotient of two differences. The relation between them is

IF f has an integral from a to b, that is, if $\int_a^b f(x)\,dx$ exists,
AND IF f has an anti-derivative, that is, there is a function F such that $dF/dx = f$,
THEN

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Are there cases where one of these exists without the other? Yes, though I'll admit that you're not likely to come across such functions without hunting through some advanced math books.

Notice an important result that follows from this equation. Differentiate both sides with respect to b

$$\frac{d}{db}\int_a^b f(x)\,dx = \frac{d}{db}F(b) = f(b)$$

and with respect to a

$$\frac{d}{da}\int_a^b f(x)\,dx = -\frac{d}{da}F(a) = -f(a)$$

Differentiating an integral with respect to one or the other of its limits results in plus or minus the integrand. Combine this with the chain rule and you can do such calculations as

$$\frac{d}{dx}\int_{x^2}^{\sin x} e^{xt^2}\,dt = e^{x\sin^2 x}\cos x - e^{x^5}2x + \int_{x^2}^{\sin x} t^2 e^{xt^2}\,dt$$

You may well ask why anyone would want to *do* such a thing, but there are more reasonable examples that show up in real situations.