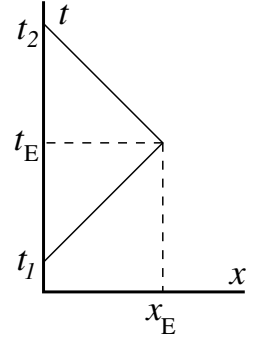


CLOCK-RADAR DETERMINATION OF SPACE AND TIME COORDINATES

We speak rather glibly about the space and time coordinates (x, t) of an event as if everyone knows just what they mean. I'll present an operational definition of these terms in a way that is designed to fit naturally with the assumptions of special relativity.

To do this, assume that a given observer has a clock and a radar set. The function of the radar set is normally to determine distances. A pulse moves with a speed c ; it is sent out at time t_1 and returns at time t_2 . The time interval for the round trip is therefore $(t_2 - t_1)$. The time interval for the trip from the spatial origin to the target is half that, or $(t_2 - t_1)/2$. The space coordinate x represents the distance traveled from the spatial origin ($x = 0$) to where the radar pulse reaches its target. The distance to the place from which it bounced is therefore:

$$\frac{c}{2}(t_2 - t_1)$$



The radar pulse reaches the target at time interval $(t_2 - t_1)/2$ after leaving at t_1 , thus it arrives at the target at time

$$t = t_1 + \frac{1}{2}(t_2 - t_1) = \frac{1}{2}(t_2 + t_1)$$

That is, just the average of the start and finish times. These measurements provide our definition of the coordinates x_E and t_E associated with the event E in which the radar pulse reaches the target.

$$x_E = \frac{c}{2}(t_2 - t_1) \quad t_E = \frac{1}{2}(t_2 + t_1)$$

DERIVATION OF THE LORENTZ TRANSFORMATION

I've given a prescription whereby an observer uses a clock and a radar set to assign coordinates to different events. If a second observer carries out the same procedure he will in general assign different sets of numbers to the same events. Since one event is thus assigned to two sets of coordinates, say (x, t) and (x', t') , I can try to find a direct algebraic relation between these so that given x and t , the values of x' and t' are easily computed.

This relation, known as the Lorentz transformation, follows automatically and rather simply now that there's a precise definition of the word "coordinate" and an understanding of the role of time dilation in the comparison of time intervals between events as measured by observers moving with respect to each other.

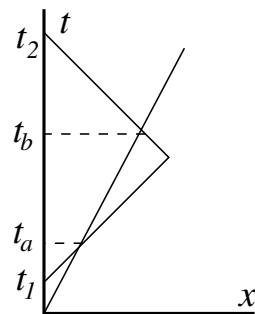
For comparison, the transformation from polar coordinates (r, θ) to cartesian coordinates (x, y) in a plane

$$x = r \cos \theta \quad y = r \sin \theta$$

and the inverse transformation is

$$r = \sqrt{x^2 + y^2} \quad \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

The first set of coordinates (x, t) are those that the stationary observer has previously assigned. He has constructed a space-time diagram such that his own motion is represented by the line $x = 0$ (the t -axis), and the x -axis is the set of points having $t = 0$. The second observer will move with a constant velocity v and will pass the stationary observer at time $t = 0$. This means that his motion has the equation $x = vt$. If at time t_1 the stationary observer sends a radar pulse in the positive x direction, the equation for the pulse (until it is reflected) will be $x = c(t - t_1)$. After reflection its motion is $x = -c(t - t_2)$, where t_2 is the time at which the reflected pulse is detected.



The second observer could go out and buy his own radar set, but why bother? The radar pulse from the first set will pass by and he can then time its flight without needing his own set. The radar pulse passes the moving observer; this event occurs when the position of the outgoing pulse, $x = c(t - t_1)$, and that of the moving observer, $x = vt$, intersect (have the same value of x and t). We will call the time that the stationary observer associates with this event t_a .

$$t = t_a \quad \text{when} \quad x = vt = c(t - t_1)$$

thus

$$t = t_a = ct_1 / (c - v)$$

The reflected pulse passes the moving observer; this event occurs when his line, $x = vt$, intersects the reflected pulse's, $x = -c(t - t_2)$. We will call the time the stationary observer associates with this event t_b .

$$t = t_b \quad \text{when} \quad x = vt = -c(t - t_2)$$

thus

$$t = t_b = ct_2 / (c + v)$$

These results can also be written

$$t_a = \frac{1 + v/c}{1 - v^2/c^2} t_1 \quad t_b = \frac{1 - v/c}{1 - v^2/c^2} t_1$$

Now, having done this much, you see what coordinates the two observers ascribe to the same event. First however you have to take into account the fact that the moving observer has a clock that is running slow (compared to the stationary observer). He therefore does not read t_a and t_b when computing coordinates. Instead, he reads the smaller numbers:

$$t'_1 = t_a \sqrt{1 - v^2/c^2} \quad t'_2 = t_b \sqrt{1 - v^2/c^2}$$

By definition the stationary observer says the coordinates of the event E are

$$x_E = \frac{c}{2}(t_2 - t_1) \quad t_E = \frac{1}{2}(t_2 + t_1)$$

and the moving observer says

$$x'_E = \frac{c}{2}(t'_2 - t'_1) \quad t'_E = \frac{1}{2}(t'_2 + t'_1)$$

Now eliminate all the intermediate variables. Get rid of $t_1, t'_1, t_2, t'_2, t_a, t_b$. To express x'_E and t'_E in terms of x_E and t_E , first rewrite t'_1 and t'_2 as

$$t'_1 = \frac{1 + v/c}{\sqrt{1 - v^2/c^2}} t_1 \quad t'_2 = \frac{1 - v/c}{\sqrt{1 - v^2/c^2}} t_2$$

then substitute these values into the equations for x'_E and t'_E . The result is

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}} \quad t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

These equations are the Lorentz transformation. (I dropped the subscript E now). The combination in the square root appears so frequently, it is worth giving it a symbol. The standard one chosen is the Greek letter gamma:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

I have assumed from the beginning that there is no intrinsic distinction between two observers moving at constant velocity with respect to each other. As a consistency check you should therefore be able to reverse the roles of the two observers in this a calculation, with v being changed to $-v$. You can do this very simply by solving the two simultaneous equations above for x and t in terms of x' and t' . The result is as it should be:

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \quad t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

These equations exhibit the peculiar feature that space and time are no longer separate concepts; they are inextricably mixed. In this sense, the set of events constituting our universe must be viewed as a single entity, space-time, rather than in terms of the two formerly unrelated ideas of space and time.

VELOCITY ADDITION

If an object is moving with a velocity $+u$ with respect to a stationary observer, and a second observer is moving with velocity $-v$, with what velocity does the second observer see the object move? The answer is not $(u + v)$, but

$$\frac{u + v}{1 + uv/c^2}$$

The proof follows directly from the Lorentz transformation equations. The equation of the object's motion is $x = ut$ (assuming it passes through the origin). The velocity is found by computing

$$\frac{x}{t} = u$$

Now find x' and t' for this line, that is the coordinates as seen by the observer moving with velocity $-v$. Using the Lorentz transform,

$$x' = \gamma(x - (-v)t) \quad t' = \gamma(t - (-vx/c^2))$$

The moving observer finds the velocity u' of the object by computing

$$\begin{aligned} u' &= \frac{x'}{t'} = \frac{\gamma(x + vt)}{\gamma(t + vx/c^2)} \\ &= \frac{(x/t) + v}{1 + vx/tc^2} = \frac{u + v}{1 + vu/c^2} \end{aligned}$$

If either u or v is small compared to c then this reduces approximately to the simpler result, $u + v$.

If $u = c$, then

$$\frac{c + v}{1 + vc/c^2} = \frac{c + v}{1 + v/c} = c$$

This equation also shows that it is impossible to add velocities in such a way as to exceed the speed of light. For example if $u = 3c/4$ and $v = 3c/4$, then

$$\begin{aligned} \frac{u + v}{1 + uv/c^2} &= \frac{\frac{3}{4}c + \frac{3}{4}c}{1 + 3^2/4^2} \\ &= \frac{24}{25}c \end{aligned}$$

so this sum remains less than the speed of light.

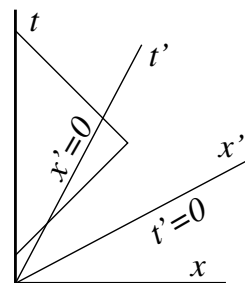
Notice that the peculiar looking extra term in the denominator of the velocity addition law, uv/c^2 , appears because of the term vx/c^2 in the equation for t' . This is the term that shows how the two observers disagree on the simultaneity of events.

SIMULTANEITY AND SPACE-TIME GEOMETRY

The stationary observer used the $x-t$ coordinate system to define simultaneity by saying that two events with the same t -coordinate are simultaneous. Since I have assumed that there is no a priori distinction between the stationary and moving observers, the moving observer should say that two events with the same t' -coordinate should be simultaneous! The x -axis is the set of simultaneous events given by the equation $t = 0$. Similarly, the moving observer takes for his space axis the set of points $t' = 0$. This is the x' -axis. From the second of the Lorentz transformation equations, this is

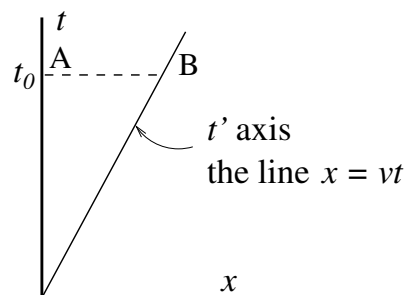
$$t' = \gamma(t - vx/c^2) = 0$$

The t -axis is the set of events occurring at the same point of space; that is, it is the graph of the stationary observer's position with the equation $x = 0$. The moving observer follows the same procedure to set up what he will call the t' -axis. He takes for this axis the graph of what he considers to be the stationary observer (himself); this is the line defined by the equation $x' = 0$, or, from the Lorentz transformations, $x - vt = 0$. Similarly, his x' -axis is the line for which $t' = 0$ at every point; from the same Lorentz equations, this is the line for which $t = (v/c^2)x$.



On the space time diagram the units of the axes were chosen so that the line $x = ct$ is at 45° (for example units of microseconds for the t -axis and units of 300 meters for the x -axis). With this choice, the line $t = (v/c^2)x$ has a slope v/c , while the t' -axis has a slope of c/v . Hence the angle between the t and t' axes is the same as the angle between the x and x' axes, both equal $\tan^{-1}(v/c)$.

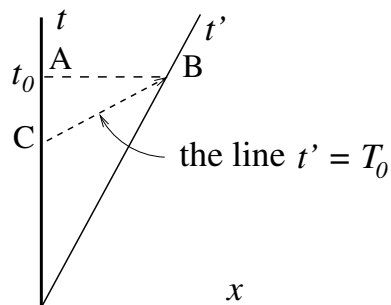
Again, ask the question: How can both observers claim that the other's clock is running slow? Start off with the stationary observer. When his clock reads t_0 , he says that *at that same time* the moving observer's clock read $t_0\sqrt{1 - v^2/c^2}$. In order to see this in terms of the space-time diagram we need only find the coordinates of the specified clocks. The stationary clock is at $x = 0$, $t = t_0$, the point A in the figure; the moving clock at this time t_0 is at position $x = vt_0$, the point B. We now ask: what does the moving clock read? This is just the definition of the t' coordinate for this event. Use the Lorentz transformation to find x' and t' when $x = vt_0$, $t = t_0$.



$$\begin{aligned} x' &= \gamma(vt_0 - vt_0) & t' &= \gamma(t_0 - v^2t_0/c^2) \\ &= 0 & &= t_0\sqrt{1 - v^2/c^2} \end{aligned}$$

The value for x' naturally appears because the moving observer claims that he is stationary and everyone else is moving. (It's a free universe.) The value for t' simply reproduces what we already know, that the moving clock has slowed.

We now ask the moving observer to do the same computation. At some time, which may as well be the point B, his clock reads T_0 ($= t_0\sqrt{1 - v^2/c^2}$). *At this same time* what does the stationary clock read? Remember now, it is the moving observer who is doing the measurements, so simultaneous to him means at the same t' -coordinate, not at the same t -coordinate; his lines of constant time appear slanted on this diagram and are parallel to the x' -axis. To determine the stationary clock's reading simultaneous with the event B, we draw a line through B representing a constant value of t' . This is the line BC in this figure.



The equation for the line BC is precisely the statement

$$t' = T_0$$

or

$$\gamma(t - vx/c^2) = T_0$$

This is a straight line on the $x - t$ diagram representing the set of events simultaneous with B. In particular, the point C has the value $x = 0$, so at this point the stationary clock satisfies

$$\gamma(t - 0) = T_0$$

or

$$t = T_0 \sqrt{1 - v^2/c^2}$$

This means that the moving observer will see the stationary clock slowed by exactly the factor $\sqrt{1 - v^2/c^2}$, thereby proving that each observer sees the other's clock running slow.

Problems:

1. Make the substitution $v/c = \tanh \omega$ in the Lorentz transformation equations. The result will look sort of like the equations for rotating coordinates, except that you get hyperbolic functions instead of the ordinary sines and cosines.

2. Show that $x^2 - c^2t^2 = x'^2 - c^2t'^2$ under the Lorentz transformation.

3. Use the same sort of $\tanh \omega$ variables for u and v in the velocity addition equation. The result should look something like the addition formula for the tangent. Look up the addition formula for $\tanh(\omega_1 + \omega_2)$ and rewrite the velocity addition formula in terms of the ω -variables.

4. The equation for constant "proper acceleration" in relativity is $x = \sqrt{k^2 + c^2t^2} - k$, where k is some constant. Graph this. Do a power series expansion of this for small t and show that it looks like $x = at^2/2$ for some a . This number a is the acceleration. Now use this equation to describe the motion of the second observer and to derive the analog of the Lorentz transformation. You'll find the algebra a bit easier if you modify this and use the equation $x = \sqrt{k^2 + c^2t^2}$ for the motion instead. The moving observer won't pass through the origin, but that's o.k. The first step you will need is to compute the time as measured by the moving observer. That is

$$t' = \int_0^t dt \sqrt{1 - v^2/c^2}, \quad v = dx/dt$$

Answer: $x = ke^{x'/k} \cosh(ct'/k) \quad ct = ke^{-x'/k} \sinh(ct'/k)$