Two small identical small spheres with mass $m$ are hung from an insulating threads of length $L$, as shown in the figure. Each sphere has the same charge $q_1 = q_2 = q$. The radius of each sphere is very small compared to the distance between them, so that they may be considered as point charges.

(a) If the distance between the centres of the two spheres is $d$ and the angle $\theta$ is small, show

$$d = C q^{2/3}$$

and determine $C$ in terms $L$, $m$ and $g$ (the constant $\epsilon_0$ is also involved).

Equation the $x$ and $y$-components of the force on the left spheres to zero

$$F_x = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{d^2} + T \sin \theta = 0 \quad F_y = -mg + T \cos \theta = 0$$

so that eliminating $T$:

$$\frac{1}{4\pi\epsilon_0} \frac{q^2}{d^2} = mg \tan \theta \approx mg \theta$$

using the approximation $\tan \theta \approx \theta$ valid for small angles. Now $\sin \theta = d/2L$ so $\theta \approx d/L$ and we have

$$\frac{1}{4\pi\epsilon_0} \frac{q^2}{d^2} = m \frac{d}{2L} \quad \text{or} \quad d^3 = \frac{q^2 L}{2\pi\epsilon_0 mg} \quad \text{or} \quad d = C q^{2/3} \quad \text{where} \quad C = \left( \frac{L}{2\pi\epsilon_0 mg} \right)^{1/3}$$

(b) If $m = 10.0g$, $L = 1.25m$ and $q = 5.00nC$, determine $d$. ($g = 9.81m/s^2$.)

Putting in numbers:

$$d = \left( \frac{2kLq^2}{mg} \right)^{1/3} = \left( \frac{2 \times 8.988 \times 10^9 \times 1.25 \times 10^{-9}C^2}{1 \times 10^{-2} \times 9.81m/s^2} \right)^{1/3} = 0.0179m = 1.79cm$$
2 A small conducting spherical shell with inner radius $a$ and outer radius $b$ is concentric with a larger conducting spherical shell with inner radius $c$ and outer radius $d$. The inner shell has a total charge $+2q$, and the outer shell has a charge $+4q$.

(a) Calculate the electric field in terms of $q$ and the distance $r$ from the common centre of the two shells.

i For $r < a$ apply Gauss’s law for a sphere radius $r$:

$$\oint \vec{E} \cdot d\vec{A} = E \oint dA = \frac{Q_{enc}}{\epsilon_0} = 0$$

since there is not enclosed charge. So $E = 0$.

ii Now for $a < r < b$, $E = 0$ since inside a conductor.

iii For $b < r < c$

$$\oint \vec{E} \cdot d\vec{A} = E 4\pi r^2 = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0} 2q$$

so that $E = \frac{1}{4\pi \epsilon_0} \frac{2q}{r^2}$.

iv For $c < r < d$, $E = 0$ since inside a conductor.

v $d < r < e$

$$\oint \vec{E} \cdot d\vec{A} = E 4\pi r^2 = \frac{Q_{enc}}{\epsilon_0} = \frac{1}{\epsilon_0} (2q + 4q)$$

so that $E = \frac{1}{4\pi \epsilon_0} \frac{6q}{r^2}$.

Graph the result for the magnitude $E$ as a function of $r$.

(b) What is the total charge on the

i inner surface of the small shell $q_1 = 0$ since otherwise would have $E \neq 0$ in inner conductor.

ii outer surface of the small shell $q_2 = 2q$ - is total charge.

iii inner surface of the large shell $q_3 = -2q$ since otherwise would have $E \neq 0$ in outer conductor.

iv outer surface of the $q_3 = 4q$ so total on outer is $4q$. large shell
3 (a) An electric charge $Q$ is distributed uniformly around a ring of radius $a$. The ring lies in the $y-z$-plane with the origin at its centre. Find the potential $V(x)$ at a point $P$ on the ring axis at a distance $x$ from the centre of the ring. (Assume the potential $V = 0$ at infinity.)

Take a small piece $ds$ on ring. Its charge is $dq = (Q/2\pi a)ds$ and produces a contribution

$$dV = \frac{1}{4\pi \varepsilon_0} \frac{dq}{r} = \frac{Q}{2\pi a} \frac{1}{4\pi \varepsilon_0} \frac{ds}{\sqrt{r^2 + x^2}}$$

Then for the total potential

$$V = \int dV = \frac{Q}{2\pi a} \frac{1}{4\pi \varepsilon_0} \frac{1}{\sqrt{a^2 + x^2}} \int ds = \frac{Q}{2\pi a} \frac{1}{4\pi \varepsilon_0} \frac{1}{\sqrt{a^2 + x^2}} 2\pi a = \frac{1}{4\pi \varepsilon_0} \frac{Q}{\sqrt{a^2 + x^2}}$$

(b) A disk with radius $R$ has a positive uniform surface charge density $\sigma$ and lies in the $y-z$-plane with the origin at its centre. Divide the disk into a set of concentric rings and use the result of (a) to calculate the electric potential, also at a distance $x$ from the centre. Show, for $x > 0$, the potential

$$V(x) = \frac{\sigma}{2\varepsilon_0} (\sqrt{x^2 + R^2} - x).$$

(You should get an integral you know how to do!) Cut out a ring between $a$ and $a + da$. This has area $2\pi ada$ and hence a charge $(Q/\pi R^2)2\pi ada = (2Q/R^2)ada$. Using the above, which now becomes an infinitesimal

$$dV = \frac{1}{4\pi \varepsilon_0} \frac{Q}{R^2} \frac{2ada}{\sqrt{a^2 + x^2}}$$

and

$$V = \int dV = \frac{1}{4\pi \varepsilon_0} \frac{Q}{R^2} \int_0^R \frac{2ada}{\sqrt{a^2 + x^2}}$$

To do the integral, write $z = a^2 + x^2$ so that $dz = 2ada$ whence

$$\int_0^R \frac{2ada}{\sqrt{a^2 + x^2}} = \int_{x^2}^{R^2 + x^2} \frac{dz}{\sqrt{z}} = 2\sqrt{z}[x^2] = 2(\sqrt{R^2 + x^2} - \sqrt{x^2}) = 2(\sqrt{R^2 + x^2} - x)$$

assuming $x$ is positive. Then since

$$\frac{1}{4\pi \varepsilon_0} \frac{Q}{R^2} \frac{2ada}{\sqrt{a^2 + x^2}} = \frac{\sigma}{4\varepsilon_0}$$

the required result follows.

(c) By symmetry, the electric field $\vec{E}$ lies in positive $x$-direction. Use the result of (b) to show the magnitude $E = \sigma/2\varepsilon_0$ when $R \gg x$.

Have

$$E = E_x = -\frac{\partial V}{\partial x} = -\frac{\sigma}{2\varepsilon_0} \frac{\partial}{\partial x} (\sqrt{x^2 + R^2} - x) = -\frac{\sigma}{2\varepsilon_0} (\frac{x}{\sqrt{x^2 + R^2}} - 1) \approx -\frac{\sigma}{2\varepsilon_0} (\frac{x}{R} - 1) \approx \frac{\sigma}{2\varepsilon_0}$$

as expected when $R \gg x$!