PHY350 Midterm Exam
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This is a **closed book** exam.

You may **not** use your notes. You may **not** use your textbook.

You may **not** discuss the exam with anyone except Professor Curtright.

Good luck!

"On my honor, I have neither received nor given aid on this exam."

Signature:

Printed Name:

Student ID Number:

Problem 1 (25 pts)

Consider the scalar and vector functions:

$$f\left(\overrightarrow{r}\right) = xy$$

$$\overrightarrow{A}\left(\overrightarrow{r}\right) = y^2\ \widehat{x} + \left(2xy + z^2\right)\ \widehat{y} + 2yz\ \widehat{z}$$

Solution: We compute the following:

$$\overrightarrow{\nabla} f = y\ \widehat{x} + x\ \widehat{y}$$

$$\nabla^2 f = 0$$

$$\overrightarrow{\nabla} \cdot \overrightarrow{A} = 2x + 2y$$

$$\overrightarrow{\nabla} \times \overrightarrow{A} = 0$$

$$\overrightarrow{\nabla} \cdot \left(f\ \overrightarrow{A}\right) = y^3 + 4x^2y + xz^2 + 2xy^2 \quad \text{Ugh!}$$

$$\nabla^2 \left(f\ \overrightarrow{A}\right) = 6xy\ \widehat{x} + \left(4x^2 + 4y^2 + 2xy\right)\widehat{y} + 4xz\ \widehat{z} \quad \text{Ugh again!!}$$

Problem 2 (25 pts)

Find the electric potential $V(z)$ a distance z above the center of a uniformly charged flat circular disk of radius R lying in the xy-plane. The total charge on the disk is Q. What does your answer give in the two extreme cases, $z \ll R$ and $z \gg R$?

What is the electric field at the same point?

Solution: Here $\sigma = \frac{Q}{\pi R^2}$.

$$\begin{aligned} V(z) &= \frac{1}{4\pi\varepsilon_0} \int_{s=0}^{s=R} \frac{\sigma\ 2\pi s ds}{\sqrt{s^2+z^2}} \\ &= \frac{1}{4\pi\varepsilon_0} \sigma\ 2\pi \left(\sqrt{R^2+z^2} - z\right) \\ &= \frac{Q}{2\pi\varepsilon_0 R^2} \left(\sqrt{R^2+z^2} - z\right) \end{aligned}$$

$$\begin{aligned} V(z) &\underset{z \ll R}{\sim} \frac{\sigma}{2\varepsilon_0} \left(R + O\left(\frac{z^2}{R}\right) - z\right) \\ V(z) &\underset{z \gg R}{\sim} \frac{Q}{2\pi\varepsilon_0 R^2} \left(z + \frac{R^2}{2z} + O\left(\frac{R^4}{z^3}\right) - z\right) = \frac{Q}{4\pi\varepsilon_0 z} + O\left(\frac{R^2}{z^3}\right) \end{aligned}$$

$$\begin{aligned} \vec{E} &= E_z\ \hat{z} \\ E_z &= -\frac{\partial}{\partial z} V(z) = \frac{Q}{2\pi\varepsilon_0 R^2} \left(1 - \frac{z}{\sqrt{R^2+z^2}}\right) \end{aligned}$$

$$\begin{aligned} E_z &\underset{z \ll R}{\sim} \frac{\sigma}{2\varepsilon_0} + O\left(\frac{z}{R}\right) \\ E_z &\underset{z \gg R}{\sim} = \frac{Q}{4\pi\varepsilon_0 z^2} + O\left(\frac{R^2}{z^4}\right) \end{aligned}$$

Problem 3 (25 pts)

Find the electric field inside a sphere which carries a charge density proportional to the distance from the center of the sphere, $\rho = kr$, for some constant k.

If the sphere has a finite radius R, what is the total charge Q? (Express Q in terms of k and R.) What is the electric field outside the sphere?

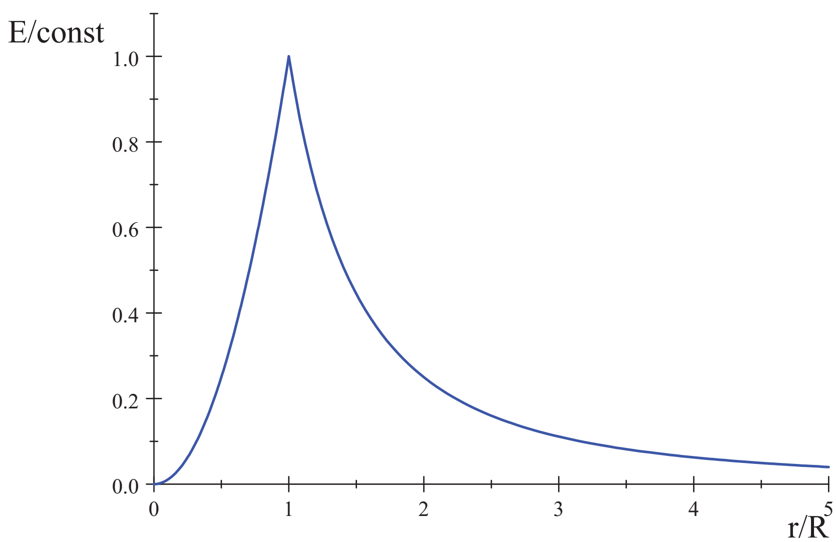
Solution: Obviously, $\overrightarrow{E} = E(r)\ \widehat{r}$

$$\begin{aligned} Q(r \leq R) &= \int_0^r 4\pi kr\ r^2 dr = \pi k r^4 \\ Q(r > R) &= \int_0^R 4\pi kr\ r^2 dr = \pi k R^4 = Q_{\text{tot}} \end{aligned}$$

Gauss's integral law for the flux then gives

$$\begin{aligned} E(r \leq R) &= \frac{\pi k r^4}{4\pi\varepsilon_0 r^2} = \frac{1}{4\varepsilon_0} k r^2 \\ E(r > R) &= \frac{\pi k R^4}{4\pi\varepsilon_0 r^2} = \frac{kR^4}{4\varepsilon_0 r^2} = \frac{Q_{\text{tot}}}{4\pi\varepsilon_0 r^2} \end{aligned}$$

Although you were not asked to do so, we plot $E(r)$.



$E/const$ versus r/R where $const = kR^2/(4\varepsilon_0)$

Problem 4 (25 pts)

Find the electric potential $V(r)$ for the spherical distribution of charge in Problem 3, for $r > R$. What is $V(r)$ for $r \leq R$?

Solution:

$$\begin{aligned} V(r > R) &= -\int_{\infty}^{r} E(r > R)\, dr \\ &= -\int_{\infty}^{r} \frac{Q_{\text{tot}}}{4\pi\varepsilon_0 r^2} dr \\ &= \frac{Q_{\text{tot}}}{4\pi\varepsilon_0}\left(-\int_{\infty}^{r} \frac{1}{r^2} dr\right) = \frac{Q_{\text{tot}}}{4\pi\varepsilon_0}\frac{1}{r} \end{aligned}$$

Alternatively,

$$V(r > R) = \frac{\pi k R^4}{4\pi\varepsilon_0}\frac{1}{r} = \frac{kR^4}{4\varepsilon_0}\frac{1}{r}$$

$$\begin{aligned} V(r \leq R) &= -\int_{\infty}^{r} E(r)\, dr \\ &= -\int_{\infty}^{R} E(r > R)\, dr - \int_{R}^{r} E(r \leq R)\, dr \\ &= V(R) - \frac{1}{4\varepsilon_0} k \int_{R}^{r} r^2 dr \\ &= \frac{Q_{\text{tot}}}{4\pi\varepsilon_0}\frac{1}{R} - \frac{1}{12\varepsilon_0} k\left(r^3 - R^3\right) \\ &= \frac{kR^4}{4\varepsilon_0}\frac{1}{R} + \frac{1}{12\varepsilon_0} kR^3 - \frac{1}{12\varepsilon_0} kr^3 \\ &= \frac{1}{3\varepsilon_0} kR^3 - \frac{1}{12\varepsilon_0} kr^3 \end{aligned}$$

Alternatively, from $k = Q_{\text{tot}}/\left(\pi R^4\right)$,

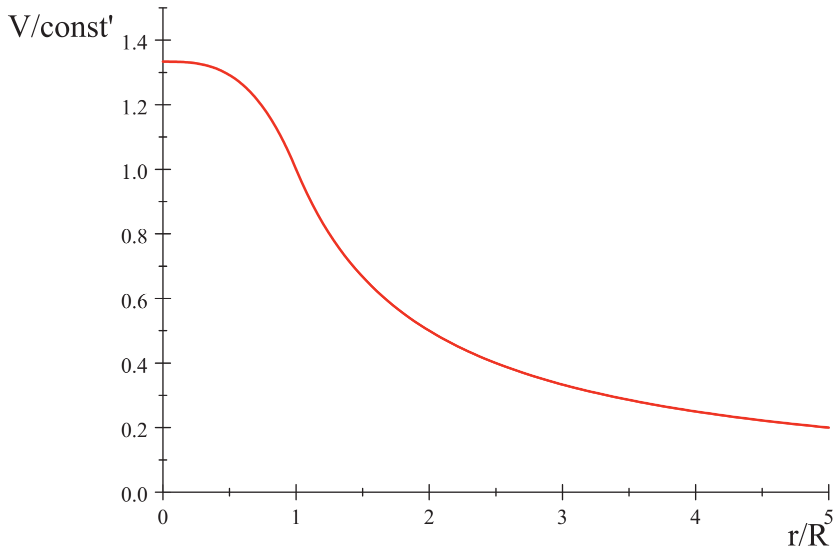
$$V(r \leq R) = \frac{Q_{\text{tot}}}{3\pi\varepsilon_0 R}\left(1 - \frac{1}{4}\frac{r^3}{R^3}\right)$$

OK, to sum up:

$$\begin{aligned} V(r > R) &= \frac{kR^3}{4\varepsilon_0}\frac{1}{r/R} \\ V(r \leq R) &= \frac{kR^3}{4\varepsilon_0}\left(\frac{4}{3} - \frac{1}{3}\frac{r^3}{R^3}\right) \end{aligned}$$

Check: $\nabla^2 V(r \leq R) = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} V(r \leq R)\right) = \frac{-1}{12\varepsilon_0} k \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr} r^3\right) = -\frac{1}{\varepsilon_0} kr = -\frac{1}{\varepsilon_0}\rho(r)$ as it should.

We also plot $V(r)$.



$V/const'$ versus r/R where $const' = kR^3/(4\varepsilon_0)$.

Problem 5 (10 Bonus points!)

A hemispherical bowl of radius $4a$ is made of conducting metal foil (very thin!) and is placed concentric with a smaller conducting sphere of radius $3a$. So, the centers of the hemisphere and sphere coincide. Put charge Q on the smaller sphere, and charge $-Q$ on the hemisphere. (Note that the charges will be distributed angularly in a non-uniform way.)

What is the potential at the point on the smaller sphere which is farthest from the edge of the hemisphere? (Note this point is equidistant from the edge, at distance $d = 5a$.)

Solution: The *potential at the center* of the small sphere is just

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{3a} - \frac{Q}{4a}\right) = \frac{Q}{48\pi\varepsilon_0 a}$$

since the charges Q and $-Q$ are all at distances $3a$ and $4a$, respectively, even though the charges are distributed angularly in some unknown way. But the potential at the center of the small spherical conductor is *the same as all points on its surface.* (It's a conductor!) Thus, the potential at the point in question is also given by this expression for V.