PHY350 Evaluation Exercise: Due Friday, 28 August 2009.

Evaluate the line integral

$$\int_{\mathcal{C}} \overrightarrow{F}\left(\overrightarrow{r}\right) \cdot d\overrightarrow{r}$$

where the vector field is

$$\overrightarrow{F}\left(\overrightarrow{r}\right) = \left(e^{-y} - ze^{-x}\right)\ \widehat{\boldsymbol{x}} + \left(e^{-z} - xe^{-y}\right)\ \widehat{\boldsymbol{y}} + \left(e^{-x} - ye^{-z}\right)\ \widehat{\boldsymbol{z}}$$

and where the curve $\mathcal{C}$ is parameterized by s, with $0 \leq s \leq 1$, as given by

$$\begin{aligned} x\left(s\right) &= \frac{\ln\left(1+s\right)}{\ln 2} \\ y\left(s\right) &= \sin\left(\frac{\pi s}{2}\right) \\ z\left(s\right) &= \frac{1-e^{s}}{1-e} \end{aligned}$$

Solution: As usual, there are hard ways and easy ways to do this, even when your vector calculus is well-honed.

If you are "thinking like a physicist" you would say, Hey! This might be a conservative force. That is to say, the vector field might be given by *a gradient*. Indeed this is true. Explicitly,

$$\overrightarrow{F}\left(\overrightarrow{r}\right) = \overrightarrow{\nabla}\Phi\left(\overrightarrow{r}\right)$$

where the gradient operation is the usual $\overrightarrow{\nabla} = \widehat{\boldsymbol{x}}\ \frac{\partial}{\partial x} + \widehat{\boldsymbol{y}}\ \frac{\partial}{\partial y} + \widehat{\boldsymbol{z}}\ \frac{\partial}{\partial z}$, and for the case at hand,

$$\Phi\left(\overrightarrow{r}\right) = xe^{-y} + ye^{-z} + ze^{-x}$$

Then the integral is just the difference of the function Φ evaluated at the end points of the curve.

$$\int_{\mathcal{C}} \overrightarrow{F}\left(\overrightarrow{r}\right) \cdot d\overrightarrow{r} = \int_{\mathcal{C}} \overrightarrow{\nabla}\Phi\left(\overrightarrow{r}\right) \cdot d\overrightarrow{r} = \Phi\left(\overrightarrow{r}\left(1\right)\right) - \Phi\left(\overrightarrow{r}\left(0\right)\right)$$

where $\overrightarrow{r}\left(1\right) = x\left(1\right)\ \widehat{\boldsymbol{x}} + y\left(1\right)\ \widehat{\boldsymbol{y}} + z\left(1\right)\ \widehat{\boldsymbol{z}} = \ \widehat{\boldsymbol{x}} + \ \widehat{\boldsymbol{y}} + \ \widehat{\boldsymbol{z}}$ and similarly $\overrightarrow{r}\left(0\right) = x\left(0\right)\ \widehat{\boldsymbol{x}} + y\left(0\right)\ \widehat{\boldsymbol{y}} + z\left(0\right)\ \widehat{\boldsymbol{z}} = 0$. Now, $\Phi\left(0\right) = 0$, and $\Phi\left(\widehat{\boldsymbol{x}} + \ \widehat{\boldsymbol{y}} + \ \widehat{\boldsymbol{z}}\right) = 1e^{-1} + 1e^{-1} + 1e^{-1} = \frac{3}{e}$. Therefore

$$\int_{\mathcal{C}} \overrightarrow{F}\left(\overrightarrow{r}\right) \cdot d\overrightarrow{r} = \frac{3}{e}$$

But, you may ask, how did I know this vector field was the gradient of something? Lucky guess?

Well, yes, actually, but there is a way to be certain that it *had* to be true. *Check the "curl" of the vector field.* That is, calculate the vector

$$\overrightarrow{\nabla}\times\overrightarrow{F}\left(\overrightarrow{r}\right)=\det\begin{pmatrix}\widehat{\boldsymbol{x}} & \widehat{\boldsymbol{y}} & \widehat{\boldsymbol{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ F_x & F_y & F_z\end{pmatrix}=\widehat{\boldsymbol{x}}\left(\frac{\partial}{\partial y}F_z-\frac{\partial}{\partial z}F_y\right)+\widehat{\boldsymbol{y}}\left(\frac{\partial}{\partial z}F_x-\frac{\partial}{\partial x}F_z\right)+\widehat{\boldsymbol{z}}\left(\frac{\partial}{\partial x}F_y-\frac{\partial}{\partial y}F_x\right)$$

When the curl vanishes, the vector field must be a gradient! One of many useful theorems from vector calculus. OK, for the case at hand

$$\begin{aligned}\frac{\partial}{\partial y}F_z-\frac{\partial}{\partial z}F_y &= \frac{\partial}{\partial y}\left(e^{-x}-ye^{-z}\right)-\frac{\partial}{\partial z}\left(e^{-z}-xe^{-y}\right)=-e^{-z}+e^{-z}=0\\ \frac{\partial}{\partial z}F_x-\frac{\partial}{\partial x}F_z &= \frac{\partial}{\partial z}\left(e^{-y}-ze^{-x}\right)-\frac{\partial}{\partial x}\left(e^{-x}-ye^{-z}\right)=-e^{-x}+e^{-x}=0\\ \frac{\partial}{\partial x}F_y-\frac{\partial}{\partial y}F_x &= \frac{\partial}{\partial x}\left(e^{-z}-xe^{-y}\right)-\frac{\partial}{\partial y}\left(e^{-y}-ze^{-x}\right)=-e^{-y}+e^{-y}=0\end{aligned}$$

All of these components vanish, hence the curl vanishes. But, say you were having a *bad* day. Suppose you computed the curl, saw it was zero, and nevertheless could not find Φ!

Well, a related theorem says that when the curl vanishes, the line integral is independent of the curve taken between the end points (note that all the functions involved here are continuous and differentiable). So, you can simplify the evaluation of the integral by taking a simpler path between the origin, $\overrightarrow{r}(0)=0$, and the opposite corner of the unit cube, $\overrightarrow{r}(1)=\widehat{\boldsymbol{x}}+\widehat{\boldsymbol{y}}+\widehat{\boldsymbol{z}}$, instead of the given curve $\mathcal{C}$. For example, take the path along the edges of the cube as follows. First along the edge $y=0=z$, so that

$$\begin{aligned}&\int_{x=0}^{x=1}\overrightarrow{F}\left(\overrightarrow{r}\right)\cdot d\overrightarrow{r}\Big|_{y=z=0}=\int_{x=0}^{x=1}F_x(x,0,0)\,dx\\ &=\int_{x=0}^{x=1}\left(e^{-y}-ze^{-x}\right)\big|_{y=z=0}\,dx=\int_{x=0}^{x=1}dx=1\end{aligned}$$

Then continue along the edge $x=1,\ z=0$, so that

$$\int_{y=0}^{y=1}\overrightarrow{F}\left(\overrightarrow{r}\right)\cdot d\overrightarrow{r}\Big|_{x=1,z=0}=\int_{y=0}^{y=1}F_y(1,y,0)\,dy$$

$$=\int_{y=0}^{y=1}\left(e^{-z}-xe^{-y}\right)\big|_{x=1,z=0}\,dy=\int_{y=0}^{y=1}\left(1-e^{-y}\right)\,dy=1+\left[e^{-y}\right|_{y=0}^{y=1}=1+\frac{1}{e}-1=\frac{1}{e}$$

Finally, continue along the edge $x=1,y=1$ to the final endpoint.

$$\int_{z=0}^{z=1}\overrightarrow{F}\left(\overrightarrow{r}\right)\cdot d\overrightarrow{r}\Big|_{x=y=1}=\int_{z=0}^{z=1}F_z(1,1,z)\,dz$$

$$=\int_{z=0}^{z=1}\left(e^{-x}-ye^{-z}\right)\big|_{x=y=1}\,dz=\int_{z=0}^{z=1}\left(\frac{1}{e}-e^{-z}\right)\,dz=\frac{1}{e}+\left[e^{-z}\right|_{z=0}^{z=1}=\frac{1}{e}+\frac{1}{e}-1=\frac{2}{e}-1$$

Adding these contributions from the three edges gives the final result.

$$\begin{aligned}
\int_{\mathcal{C}} \overrightarrow{F}\left(\overrightarrow{r}\right)\cdot d\overrightarrow{r} &= \int_{x=0}^{x=1} \overrightarrow{F}\left(\overrightarrow{r}\right)\cdot d\overrightarrow{r}\Big|_{y=z=0} + \int_{y=0}^{y=1} \overrightarrow{F}\left(\overrightarrow{r}\right)\cdot d\overrightarrow{r}\Big|_{x=1,z=0} + \int_{z=0}^{z=1} \overrightarrow{F}\left(\overrightarrow{r}\right)\cdot d\overrightarrow{r}\Big|_{x=y=1} \\
&= 1+\frac{1}{e}+\frac{2}{e}-1=\frac{3}{e}
\end{aligned}$$

in agreement with the previous answer.

Finally, say you were having a *really* bad day, and you could think of nothing simpler than to slug your way through the line integral along the original curve. Well, in that case we first write

$$\overrightarrow{F}\cdot d\overrightarrow{r}=\overrightarrow{F}\cdot\frac{d\overrightarrow{r}}{ds}ds$$

where

$$\begin{aligned}
\frac{d\overrightarrow{r}}{ds} &= \frac{dx\left(s\right)}{ds}\,\widehat{\boldsymbol{x}}+\frac{dy\left(s\right)}{ds}\,\widehat{\boldsymbol{y}}+\frac{dz\left(s\right)}{ds}\,\widehat{\boldsymbol{z}} \\
\frac{dx\left(s\right)}{ds} &= \frac{1}{\ln 2}\frac{d}{ds}\ln\left(1+s\right)=\frac{1}{\ln 2}\frac{1}{1+s} \\
\frac{dy\left(s\right)}{ds} &= \frac{d}{ds}\sin\left(\frac{\pi s}{2}\right)=\frac{\pi}{2}\cos\left(\frac{\pi s}{2}\right) \\
\frac{dz\left(s\right)}{ds} &= \frac{1}{1-e}\frac{d}{ds}\left(1-e^{s}\right)=\frac{1}{1-e}\left(-e^{s}\right)
\end{aligned}$$

Then we have

$$\begin{aligned}
\overrightarrow{F}\cdot\frac{d\overrightarrow{r}}{ds} &= F_x\frac{dx\left(s\right)}{ds}+F_y\frac{dy\left(s\right)}{ds}+F_z\frac{dz\left(s\right)}{ds} \\
&= \left(e^{-y}-ze^{-x}\right)\frac{dx\left(s\right)}{ds}+\left(e^{-z}-xe^{-y}\right)\frac{dy\left(s\right)}{ds}+\left(e^{-x}-ye^{-z}\right)\frac{dz\left(s\right)}{ds} \\
&= \left(e^{-\sin\left(\frac{\pi s}{2}\right)}-\frac{1-e^{s}}{1-e}e^{-\frac{\ln(1+s)}{\ln 2}}\right)\frac{1}{\ln 2}\frac{1}{1+s} \\
&\quad +\left(e^{-\frac{1-e^{s}}{1-e}}-\frac{\ln\left(1+s\right)}{\ln 2}e^{-\sin\left(\frac{\pi s}{2}\right)}\right)\frac{\pi}{2}\cos\left(\frac{\pi s}{2}\right) \\
&\quad +\left(e^{-\frac{\ln(1+s)}{\ln 2}}-\sin\left(\frac{\pi s}{2}\right)e^{-\frac{1-e^{s}}{1-e}}\right)\frac{1}{1-e}\left(-e^{s}\right)
\end{aligned}$$

Ugh! Now you see what I meant by having a *REALLY* bad day. Each of these terms must be integrated from $s=0$ to $s=1$.

Sadly, the individual terms are *not* elementary integrals. But, collectively the integration of all the terms can be done just because the combination is *a total derivative* with respect to s. Like so

$$\begin{aligned}
\overrightarrow{F}\cdot\frac{d\overrightarrow{r}}{ds} &= \left(e^{-y(s)}-z\left(s\right)e^{-x(s)}\right)\frac{dx\left(s\right)}{ds}+\left(e^{-z(s)}-x\left(s\right)e^{-y(s)}\right)\frac{dy\left(s\right)}{ds}+\left(e^{-x(s)}-y\left(s\right)e^{-z(s)}\right)\frac{dz\left(s\right)}{ds} \\
&= \frac{d}{ds}\left(\,x\left(s\right)e^{-y(s)}+y\left(s\right)e^{-z(s)}+z\left(s\right)e^{-x(s)}\,\right)
\end{aligned}$$

Admittedly, this is tantamount to finding the function Φ as given previously, but sometimes it is useful to go through this extra step to help you get Φ

However, even if you cannot do the integrals analytically, in terms of known functions, you can always do them *numerically*. Explicitly, to five decimal places, I find

$$\int_0^1 \left(e^{-\sin\left(\frac{\pi s}{2}\right)} - \frac{1-e^s}{1-e} e^{-\frac{\ln(1+s)}{\ln 2}}\right) \frac{1}{\ln 2} \frac{1}{1+s} ds = 0.410\,35$$

$$\int_0^1 \left(e^{-\frac{1-e^s}{1-e}} - \frac{\ln(1+s)}{\ln 2} e^{-\sin\left(\frac{\pi s}{2}\right)}\right) \frac{\pi}{2} \cos\left(\frac{\pi s}{2}\right) ds = 0.548\,28$$

$$\int_0^1 \left(e^{-\frac{\ln(1+s)}{\ln 2}} - \sin\left(\frac{\pi s}{2}\right) e^{-\frac{1-e^s}{1-e}}\right) \frac{1}{1-e} \left(-e^s\right) ds = 0.145\,01$$

so then

$$\int_0^1 \left(e^{-\sin\left(\frac{\pi s}{2}\right)} - \frac{1-e^s}{1-e} e^{-\frac{\ln(1+s)}{\ln 2}}\right) \frac{1}{\ln 2} \frac{1}{1+s} ds + \int_0^1 \left(e^{-\frac{1-e^s}{1-e}} - \frac{\ln(1+s)}{\ln 2} e^{-\sin\left(\frac{\pi s}{2}\right)}\right) \frac{\pi}{2} \cos\left(\frac{\pi s}{2}\right) ds + \int_0^1 \left(e^{-\frac{\ln(1+s)}{\ln 2}} - \sin\left(\frac{\pi s}{2}\right) e^{-\frac{1-e^s}{1-e}}\right) \frac{1}{1-e} \left(-e^s\right) ds = 1.\,103\,6$$

Now lo and behold, to five decimal places, $\frac{3}{e} = 1.\,103\,6$, so we are again in agreement with the first result given above.

For this numerical work on my computer, I have used a software package called *Maple*. Similarly, I could have used *Mathematica*, or *MuPAD*, or any of several other readily available programs. A simple internet search for these program names will lead you to websites where they are available, often for *free* on a trial basis (30 days or more). You might enjoy taking a look.