Extrinsic Curvature, Polyakov, Weyl, and Einstein

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In the mid-1980s Polyakov [1] suggested that "fine structure" could be added to string dynamics by including an extrinsic curvature term in the world sheet action.

$$A = \int_{M_2} \left(T + S\ K_{\alpha\beta} K^{\alpha\beta} \right) \sqrt{\det g_{\alpha\beta}}\ d^2x$$

where T is the tension, $g_{\alpha\beta}$ is the induced metric, S is the coefficient of the term that "stiffens" the string, i.e. adds "rigidity" to the world sheet, and where

$$K_{\alpha\beta} = -\widehat{n} \cdot \frac{\partial^2 \overrightarrow{r}}{\partial x^\alpha \partial x^\beta}$$

is the second form that encodes the extrinsic curvature of the world-sheet. In this last expression, $\overrightarrow{r}$ is a point on the world-sheet and $\widehat{n}$ is a local unit normal to the sheet. The local dynamics (i.e. equations of motion) of the string are modified significantly by this rigidity. In collaboration with Ghassan Ghandour, Charles Thorn, and Cosmas Zachos, I studied the interesting effects this would have on Regge trajectories [2], but that is *not* the subject of this talk.

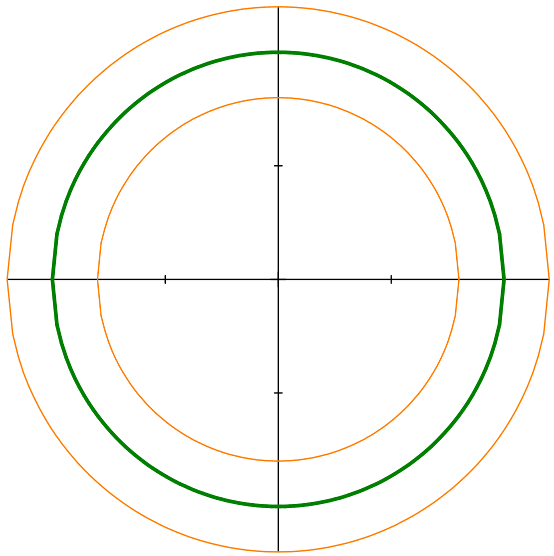
Soon after studying the modified Regge trajectories, I went on sabbatical in the fall of 1986, to visit the ITP at Stony Brook. It occurred to me while there that perhaps Polyakov's rigidity term could be obtained just as it is in structural engineering by distributing material transverse to the world sheet. I remind you that this is how linear beams are stiffened to resist bending.

[1] Talk given 12 May 2017 at Fontbonne University.

But a transverse thickening of the string world-sheet would just turn it into a membrane world-volume. So I spent some time thinking about membranes, supermembranes, and all that. I gave several talks about my work, including a weekly lecture course at Yale in the first half of 1987, and I wrote a couple of papers on the subject [3]. The last of these was based on a lecture I gave at a conference in Copenhagen during the fall of 1987 while I was a visiting scientist at CERN.

Unfortunately, my original idea to stiffen the world-sheet did *not* work. Here's why. It takes only a simple calculation to understand what is going on. For visualization purposes, I work in Euclidean space.

Consider a spherical world-sheet thickened to become a world-volume, namely, the volume between two concentric spheres of radii $r - \varepsilon$ and $r + \varepsilon$, shown here in cross-section.[2]



[2]This simple calculation can be found in the physics literature [4]. I thank Eduardo Guendelman for bringing this paper to my attention.

The volume of the region between the concentric orange spheres is

$$\begin{aligned} V &= \frac{4\pi}{3}\left((r+\varepsilon)^3 - (r-\varepsilon)^3\right) \\ &= 4\pi r^2 \times 2\varepsilon\left(1 + \frac{\varepsilon^2}{3r^2}\right) \end{aligned}$$

which may be written in a suggestive way as

$$V = 4\pi r^2 \int_{-\varepsilon}^{\varepsilon} \det \begin{pmatrix} 1 + y/r & 0 \\ 0 & 1 + y/r \end{pmatrix} dy$$

So, thinking of this as the world-volume of a membrane with surface tension T_2, the membrane action would be

$$A_2 = T_2 V = 4\pi r^2 \times 2\varepsilon T_2\left(1 + \frac{\varepsilon^2}{3r^2}\right)$$

In fact this is just the modified action of a string, whose world-sheet is the sphere of radius r with effective tension

$$T = 2\varepsilon T_2$$

and which includes a curvature term given by

$$C = \frac{T\varepsilon^2}{3r^2}$$

At first glance, this would seem to be the Polyakov action with $S \propto T\varepsilon^2$, so did this simple thickening of the world-sheet actually do the job to induce rigidity?

Well, no. Close perhaps, but no cigar! This curvature term actually involves the intrinsic scalar curvature

$$R = (K_\alpha{}^\alpha)^2 - K_{\alpha\beta}K^{\alpha\beta}$$

and *not* just $K_{\alpha\beta}K^{\alpha\beta}$. Viewed as a modified string action, the integral of the intrinsic scalar curvature over the world-sheet would give the topological Euler characteristic, and would not modify the local dynamics of the string.

To confirm this, instead of a sphere consider a right circular cylinder of length L and radius r. As a surface, this has *no* intrinsic curvature, i.e. $R = 0$, since one of the radii of curvature is infinite, although it does have extrinsic curvature $K_{\alpha\beta}K^{\alpha\beta} = 1/r^2$. Repeating the previous simple calculation for concentric cylinders now gives a volume between the cylinders of

$$\begin{aligned} V &= \pi\left((r+\varepsilon)^2 - (r-\varepsilon)^2\right)L \\ &= 2\pi r L \times 2\varepsilon \\ &= 2\pi r L \int_{-\varepsilon}^{\varepsilon} \det\begin{pmatrix} 1 + y/r & 0 \\ 0 & 1+0 \end{pmatrix} dy \end{aligned}$$

Thus the action for this membrane world-volume would be

$$A_2 = T_2 V = 2\pi r L \times 2\varepsilon T_2 \left(1 + \frac{0}{r^2}\right)$$

Viewed as a string with a cylindrical world-sheet, this again has an effective tension $T = 2\varepsilon T_2$, but now there is no induced curvature term.

I considered higher dimensional generalizations, but the story was always the same, when computed as above: No induced Polyakov curvature term. Only intrinsic curvature terms are obtained.

But later on, Ulf Lindstrom rose to the challenge [5] and pointed out how Polyakov's extrinsic curvature term *could* be obtained by embedding the membrane in an extra spatial dimension, and then taking a limit where both the size of the extra dimension and the thickness of the membrane world-volume simultaneously went to zero. The calculation above is *not* applicable in that case, for reasons that I will indicate below (if they are not already obvious). I have not checked Ulf's calculation, but I have no reason to doubt it. While there was a bit of regret on my part for not having done the calculation that Ulf did, I took solace in the fact that he gave me some credit for the essential idea. As he said:

> Polyakov introduces the rigidity term in the string action in an attempt to find a string that will correspond to QCD in some limit. In this context it seems natural to try to interpret the rigidity term as a "memory" of additional dimensions that otherwise play no role in this limit [#1]. In the present letter we show that the rigidity term in Polyakov's rigid string action formally may be obtained via compactification from a higher dimensional membrane.
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> [#1] This point of view has been emphasized e.g. in ref. [9].
>
> [9] T.L. Curtright, private communication.

Although, it would have been even more comforting had he cited one of my published papers on the subject.

Anyway, how does it go in higher dimensions? If you embed a curved n dimensional submanifold into a higher N dimensional Euclidean space then for a "tubular" embedding, analogous to what I did above for concentric spheres and cylinders, the volume element in the full space is given by

$$d^N X = \sqrt{\det g_{\alpha\beta}}\; d^n x \;\; \det\left(1 + \sum_{a=1}^{N-n} y^a \; \mathbb{K}^{(a)}\right) \; d^{N-n} y$$

where $g_{\alpha\beta}$, for $\alpha, \beta = 1, \cdots, n$, is the metric on the submanifold, $\overrightarrow{y}$ are points in the $N - n$ dimensional ambient space surrounding the submanifold, and

$$\mathbb{K}^{(a)}_{\alpha\beta} = -g^{\alpha\gamma} \; \widehat{n}^{(a)} \cdot \frac{\partial^2 \overrightarrow{r}}{\partial x^\gamma \partial x^\beta}$$

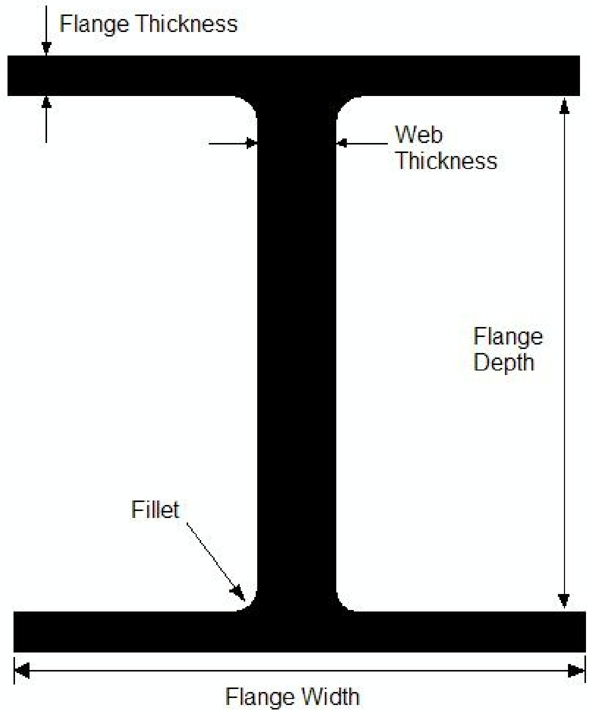
Here $\overrightarrow{r}$ is a point in the submanifold, and $\widehat{n}^{(a)}$ for $a = 1, \cdots, N - n$ are all the locally orthogonal unit normals to the submanifold. This result was obtained by Weyl, although it is just elementary calculus on manifolds. It is known as the tube volume formula. Weyl went farther to point out that integrating the $\overrightarrow{y}$ over an isotropic ambient space gives only combinations of the 2nd form matrix that can be expressed in terms of intrinsic curvatures. All this mathematics is discussed in books, e.g. [6]. It has also been discussed in the physics literature [7][3], most recently as a scheme to induce Einstein gravity on four-dimensional spacetime [8], but always following the same logic that I described above for strings and membranes.

[3] I thank Ricardo Troncoso for pointing out this paper.

For simplicity, consider just $N = n + 1$, as I did in the string/membrane case. Then in general the effective volume measure on the submanifold for a tubular embedding is

$$\sqrt{\det g_{\alpha\beta}}\ d^n x \int_{-\varepsilon}^{\varepsilon} \rho\left(\overrightarrow{r}, y\right) \det\left(1 + y\ \mathbb{K}_{n\times n}\right)\ dy$$

where $\rho\left(\overrightarrow{r}, y\right)$ describes a suitable distribution in the ambient space at each submanifold point, $\overrightarrow{r}$. Previously, for the string/membrane case, I used constant ρ. But if God were a structural engineer (with a gambling problem?) he might have been enticed by something more like the following, with the submanifold at the center of the "web"[4] and with y indicating the vertical position, say.



[4]That's right, we might have been living on the web all along, well before the internet was invented.

In any case, if you take ρ to be an even function of y, then the y integration produces only intrinsic curvature combinations. Suppressing the $\overrightarrow{r}$ dependence for convenience,

$$\int_{-\varepsilon}^{\varepsilon} \rho_{even}(y) \det\left(1 + y\ \mathbb{K}_{n\times n}\right)\ dy = \int_{-\varepsilon}^{\varepsilon} \rho_{even}(y) + \frac{1}{2}\ R \int_{-\varepsilon}^{\varepsilon}\ y^2 \rho_{even}(y)\, dy + \cdots$$

This follows most easily from an eigenvalue expansion of the determinant.

$$\det\left(1 + y\ \mathbb{K}_{n\times n}\right) = 1 + y\left(\sum_{i=1}^{n} \kappa_i\right) + y^2\left(\sum_{\substack{i,j=1 \\ j>i}}^{n} \kappa_i\kappa_j\right) + \cdots + y^n \prod_{i=1}^{n} \kappa_i$$

where in diagonal form, at each submanifold point,

$$\mathbb{K}_{n\times n} = \begin{pmatrix} \kappa_1\left(\overrightarrow{r}\right) & & & \\ & \kappa_2\left(\overrightarrow{r}\right) & & \\ & & \ddots & \\ & & & \kappa_n\left(\overrightarrow{r}\right) \end{pmatrix}$$

It is well-known [9] that the even terms in the eigenvalue expansion of the above determinant can be expressed in terms of intrinsic curvature polynomials.

Meanwhile, back in physics ...

Obviously, the Weyl formula is not reliable, and probably breaks down, when $y \approx \min(1/\kappa)$, since the volume measure is required to be positive definite. This is the mathematical nicety that allowed Ulf Lindstrom to obtain a Polyakov term, I believe.

The leading term, $\int_{-\varepsilon}^{\varepsilon} \rho(y)\, dy$, would represent an effective cosmological "constant" Λ — actually constant only if there is no $\overrightarrow{r}$ dependence in ρ.

The second moment, $\frac{1}{2}\ R \int_{-\varepsilon}^{\varepsilon} y^2 \rho(y)\, dy$, is effectively the Einstein-Hilbert density, where the integral over y gives the Newton/Einstein "constant" — again strictly constant only if there is no $\overrightarrow{r}$ dependence in ρ. Note that for positive definite ρ the Λ and R terms must have the same sign. (Wikipedia says they do **not**, in the real world, but one has to be careful about conventions.) All the other even terms in the expansion must also have the same sign, unless of course ρ flips sign for some values of y.

In principle, some $\overrightarrow{r}$ dependence in $\rho(\overrightarrow{r}, y)$ could account for the anomalous galaxy and cluster rotation curves that are cited as evidence for dark matter. But so far, I have not been able to carefully check the numbers.

From the numerical value of the Einstein constant and the present experimental value of Λ, one estimates that the "flange depth" is approximately the size of the observable universe.[5] Oh Lord!

Finally, if you want a technical problem to investigate, I have not yet found in the literature the extension of Weyl's tube formula to superspace. Of course, the replacement det $\rightarrow$ sdet is a no-brainer, but the correct replacement of $\mathbb{K}$ in the above formulas is more challenging. The result given in my paper with Peter van Nieuwenhuizen [3] seems to be the correct one, even if it leads to a supergravity action of the form proposed by Arnowit and Nath [10], rather than directly to the forms found by Ferrara, Freedman, and van Nieuwenhuizen [11], or by Deser and Zumino [12].

[5] As noted by David Fairlie, this brings to mind Mach's principle.

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References

[1] A M Polyakov

[2] T L Curtright, G I Ghandour, C B Thorn, and C K Zachos
T L Curtright, G I Ghandour, and C K Zachos

[3] T L Curtright and P van Nieuwenhuizen
T L Curtright

[4] C.Barrabes, B.Boisseau, and M.Sakellariadou

[5] U Lindstrom

[6] A Gray, *Tubes*, 2nd edition (2004) Birkhäuser

[7] S Willison

[8] O Alvarez
O Alvarez and M Haddad

[9] See any good textbook on differential geometry.

[10] R L Arnowitt and P Nath

[11] D Z Freedman, P van Nieuwenhuizen, and S Ferrara

[12] S Deser and B Zumino