Massive Dual Gravity Revisited

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Talk at U of Kansas, 26 September 2019

Abstract

I will describe a highly speculative model of gravity as a massive, pure spin 2 field, which is "dual" to the usual description in terms of a spacetime metric tensor.

In the dual description, for weak fields, the metric emerges as the field strength of an underlying fundamental field. More generally, if the gravitational field is not weak, the metric emerges as a nonlinear mixture involving the energy momentum tensor.

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*A pre-emptive answer to the question: "Why abbreviate Thomas as Thom?"

For me it has nothing to do with Thom McAn. But it does go back to my youth. In the DC universe Thom Kallor is *Star Boy*, a member of the Legion of Super-Heroes living in the next millennium. He was born to astronomer parents on an observation satellite orbiting the planet Xanthu.

Star Boy is *able to increase the mass of an object*, but only temporarily. It seems appropriate to mention that here, given the subject of this talk.

Here is the basic mechanism behind massive duality.

Consider a scalar field ϕ and antisymmetric tensor $V_{\lambda\mu\nu}$ in 4D.
Let

$$\mathcal{L}\left[\phi, V\right] = -\frac{1}{2}\, m^2\phi^2 + \frac{1}{6}\, m\varepsilon^{\kappa\lambda\mu\nu}\phi\, \partial_\kappa V_{\lambda\mu\nu} + \frac{1}{12}\, m^2 V_{\lambda\mu\nu}V^{\lambda\mu\nu}$$

Classically, the field equations are

$$m\phi = \frac{1}{6}\, \varepsilon^{\kappa\lambda\mu\nu}\partial_\kappa V_{\lambda\mu\nu}\ , \quad mV^{\lambda\mu\nu} = \varepsilon^{\kappa\lambda\mu\nu}\partial_\kappa\phi$$

Note the interchange: field $\leftrightarrow$ field strength.

Also note that $\partial_\lambda V^{\lambda\mu\nu} = 0$ for $m \neq 0$.

It follows that

$$
\begin{aligned}
m^2\phi &= -\frac{1}{6}\,\varepsilon^{\kappa\lambda\mu\nu}\varepsilon_{\lambda\mu\nu\rho}\,\partial_\kappa\partial^\rho\phi = -\frac{1}{6}\,\delta^{\kappa\lambda\mu\nu}_{\rho\lambda\mu\nu}\,\partial_\kappa\partial^\rho\phi = -\Box\phi \\
m^2V^{\lambda\mu\nu} &= \frac{1}{6}\,\varepsilon^{\kappa\lambda\mu\nu}\varepsilon_{\alpha\beta\gamma\delta}\,\partial_\kappa\partial^\alpha V^{\beta\gamma\delta} = -\frac{1}{6}\,\delta^{\kappa\lambda\mu\nu}_{\alpha\beta\gamma\delta}\,\partial_\kappa\partial^\alpha V^{\beta\gamma\delta} = -\Box V^{\lambda\mu\nu}
\end{aligned}
$$

So, both ϕ and V are fields of mass m.

$$
\left(\Box + m^2\right)\phi = 0\ , \quad \left(\Box + m^2\right)V^{\lambda\mu\nu} = 0
$$

Either field describes a massive, spinless particle.

Alternatively, in the path integral, complete the square and integrate out ϕ to find a massive V theory:

$$\mathcal{L}\left[V\right]=\frac{1}{36}\,\left(\varepsilon^{\kappa\lambda\mu\nu}\partial_{\kappa}V_{\lambda\mu\nu}\right)^{2}+\frac{1}{12}\,m^{2}V_{\lambda\mu\nu}V^{\lambda\mu\nu}$$

Or integrate out $V_{\lambda\mu\nu}$ to find a massive ϕ theory:

$$\mathcal{L}\left[\phi\right]=\frac{1}{2}\left(\partial_{\kappa}\phi\right)\left(\partial^{\kappa}\phi\right)-\frac{1}{2}\,m^{2}\phi^{2}$$

Either way you obtain again a description of a massive, spinless field.

Let's move on to spin 2.

For a symmetric tensor field, $h_{\mu\nu} = h_{\nu\mu}$, recall that

$(\Box + m^2)\, h_{\mu\nu} = \kappa\, \Theta_{\mu\nu}$ for $\Theta_{\mu\nu} = \Theta_{\nu\mu}$ & $\partial^\mu \Theta_{\mu\nu} = 0$

produces spin 2 $\oplus$ spin 0 radiation if $\Theta_\alpha^{\ \alpha} \neq 0$ but

$(\Box + m^2)\, h_{\mu\nu} = \kappa\, \Theta_{\mu\nu} + \frac{1}{3m^2} \kappa \left(\eta_{\mu\nu} \Box - \partial_\mu \partial_\nu \right) \Theta$

produces *only* spin 2 radiation even if $\Theta_\alpha^{\ \alpha} \neq 0$

For the 1st option above, see Freund, Maheshwari, & Schonberg (1968) while for the 2nd option, see Ogievetski and Polubarinov (1965) or more recently, de Rham, Gabadadze, and Tolley (2011).

This talk concerns the dual form of the OP-dRGT model, first proposed by TLC & PGOF in 1980 without reference to OP.

Massive gravity

— as usually presented [1] — starting from the massless theory in the weak field limit.

In four-dimensional spacetime, where

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \ , \quad h = h_{\mu}^{\ \mu} \ , \quad \eta_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1) \ ,$$

for weak fields with $\|h_{\mu\nu}\| \ll 1$ Einstein's equations become

$$\Box h_{\mu\nu} - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta h_{\alpha\beta} - \eta_{\mu\nu} \Box h + \partial_\mu \partial_\nu h = \kappa \Theta_{\mu\nu} + O\left(h_{\alpha\beta}^2\right)$$

where $\Box = \frac{1}{c^2}\partial_t^2 - \nabla^2$ is the d'Alembertian, and $\Theta_{\mu\nu}$ represents the energy-momentum for everything else in the world. The LHS is manifestly divergenceless.

That is to say, in the linear approximation,

$$R_{\mu\nu} = \frac{1}{2}\left(\Box h_{\mu\nu} - \partial_\mu \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha h_{\alpha\mu} + \partial_\mu \partial_\nu h\right) \ , \qquad R = \Box h - \partial^\alpha \partial^\beta h_{\alpha\beta}$$

so that Einstein's equations,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\, g_{\mu\nu} R = \frac{1}{2}\, \kappa \Theta_{\mu\nu} \ .$$

with $\kappa = 16\pi G/c^4$, then lead to the previous linearized field equations.

Those field equations are slightly simplified if $h_{\mu\nu}$ is replaced by the "trace-reversed" field $\hbar_{\mu\nu}$.[1]

$$\hbar_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu} h \ , \quad h_{\mu\nu} = \hbar_{\mu\nu} - \frac{1}{2}\, \eta_{\mu\nu} \hbar \ , \quad h = -\hbar \ .$$

The result is

$$\Box \hbar_{\mu\nu} - \partial_\mu \partial^\alpha \hbar_{\alpha\nu} - \partial_\nu \partial^\alpha \hbar_{\alpha\mu} + \eta_{\mu\nu} \partial^\alpha \partial^\beta \hbar_{\alpha\beta} = \kappa \Theta_{\mu\nu}$$

[1] My apologies to Planck, but this notation was just too good to resist.

Even better, if the "harmonic gauge condition" is chosen,

$$0 = \partial^\alpha \bar{h}_{\alpha\mu} = \partial^\alpha h_{\alpha\mu} - \frac{1}{2}\,\partial_\mu h$$

then the field equations reduce to something a good undergraduate can solve, given $\Theta_{\mu\nu}$.

$$\Box \bar{h}_{\mu\nu} = \kappa \Theta_{\mu\nu}$$

In these gauges the Ricci and scalar curvatures, and the Einstein tensor, are just

$$R_{\mu\nu} = \frac{1}{2}\,\Box h_{\mu\nu}\ , \quad R = \frac{1}{2}\,\Box h\ , \quad G_{\mu\nu} = \frac{1}{2}\,\Box h_{\mu\nu} - \frac{1}{4}\,\eta_{\mu\nu}\Box h$$

In 1939 Markus Fierz and Wolfgang Pauli determined the "correct" mass term for the theory. They added $\frac{1}{4}m^2\left(h^2 - h^{\mu\nu}h_{\mu\nu}\right)$ to the Lagrangian, hence $m^2\left(h_{\mu\nu} - \eta_{\mu\nu}h\right)$ to the previous linearized field equations.[2]

Adding the most general mass term, namely, $m^2 h_{\mu\nu} - f\ m^2\eta_{\mu\nu}h$ for constant f, the linearized Einstein equations become

$$\left(\Box + m^2\right)h_{\mu\nu} - \partial_\mu\partial^\alpha h_{\alpha\nu} - \partial_\nu\partial^\alpha h_{\alpha\mu} + \eta_{\mu\nu}\partial^\alpha\partial^\beta h_{\alpha\beta} - \eta_{\mu\nu}\left(\Box + f\ m^2\right)h + \partial_\mu\partial_\nu h = \kappa\Theta_{\mu\nu} + O\left(h^2_{\alpha\beta}\right)$$

[2] I am told that data from gravitational wave detections require $mc^2 < 1 \times 10^{-28}$ MeV, to be compared to results from weak gravitational lensing, i.e. $mc^2 < 1 \times 10^{-37}$ MeV, and to more traditional electromagnetic experiments which give limits for the photon mass, i.e. $m_\gamma c^2 < 1 \times 10^{-24}$ MeV. But the bounds on m are *not* enough to rule out a Compton wave length for cosmological distance scales, since $mc^2 \sim 1 \times 10^{-39}$ MeV for the Hubble distance.

Under the assumption that $\partial^\mu \Theta_{\mu\nu} = 0$, taking the divergence of these field equations gives

$$\left(\Box + m^2\right) \partial^\mu h_{\mu\nu} - \Box \partial^\alpha h_{\alpha\nu} - \partial_\nu \partial^\alpha \partial^\mu h_{\alpha\mu} + \partial_\nu \partial^\alpha \partial^\beta h_{\alpha\beta} - \partial_\nu \left(\Box + f\ m^2\right) h + \Box \partial_\nu h = 0 + O\left(h_{\alpha\beta}^2\right)$$

That is to say, to the order considered, we obtain the condition

$$m^2 \left(\partial^\mu h_{\mu\nu} - f\ \partial_\nu h\right) = 0 + O\left(h_{\alpha\beta}^2\right)$$

Note this is *not* the harmonic gauge condition for the Fierz-Pauli choice $f = 1$.

On the other hand, taking the trace of the field equations now gives

$$-\left(2\Box + (4f-1)\,m^2\right) h + 2\partial^\alpha\partial^\beta h_{\alpha\beta} = \kappa\Theta + O\left(h_{\alpha\beta}^2\right)$$

where $\Theta = \Theta_\mu{}^\mu$. When combined with the previous condition, this becomes

$$2\,(1-f)\,\Box h + (1-4f)\,m^2 h = \kappa\Theta + O\left(h_{\alpha\beta}^2\right)$$

and the trace h does not drop out, in general. Therefore, to the order considered, the trace h vanishes or decouples if and only if the energy-momentum tensor is traceless.

But more importantly, instead of $f = 1$ as in the Fierz-Pauli combination, any other choice for this relative coefficient in the mass terms will result in independent *propagation* of the trace h. That is to say, a sixth degree of freedom can exist as radiation, in addition to the five expected for pure spin 2, and that extra scalar degree of freedom can be produced by energy-momentum if $\Theta \neq 0$.

Massive gravity reconsidered

— from a dual point of view [2] — rather than the usual one.

Instead of the symmetric rank two tensor, $g_{\mu\nu} = g_{\nu\mu}$, in four spacetime dimensions there is another way to describe a massive spin 2 particle using a rank *three* tensor field, $T_{[\lambda\mu]\nu}$. The tensor must have the symmetries

$$T_{[\lambda\mu]\nu} = -T_{[\mu\lambda]\nu} \ , \quad T_{[\lambda\mu]\nu} + T_{[\mu\nu]\lambda} + T_{[\nu\lambda]\mu} = 0$$

corresponding to those of the Young diagram .

Note: These are the same symmetries as those of the Lanczos tensor, but in general, $T_{[\lambda\mu]\nu}$ is *not* the same as Cornelius' construction.

In this case the “free” field equations follow from the Lagrangian

$$\mathcal{L} = F_{[\lambda\mu\nu]\rho}\, F^{[\lambda\mu\nu]\rho} - 3F_{[\mu\nu]}\, F^{[\mu\nu]} - 3m^2\left(T_{[\lambda\mu]\nu}T^{[\lambda\mu]\nu} - 2T_\lambda T^\lambda\right)$$

where the trace of the field is $T_\lambda = g^{\mu\nu}T_{[\lambda\mu]\nu}$, and the field strength and its trace are

$$F_{[\lambda\mu\nu]\rho} \equiv \partial_\lambda T_{[\mu\nu]\rho} + \partial_\mu T_{[\nu\lambda]\rho} + \partial_\nu T_{[\lambda\mu]\rho}$$

$$F_{[\mu\nu]} \equiv g^{\lambda\rho}F_{[\lambda\mu\nu]\rho}\ ,$$

The actual field equation resulting from the action for $\mathcal{L}$ is somewhat complicated, with six terms, as was the case for $h_{\mu\nu}$.

But in flat spacetime it all boils down to equations in the standard Fierz-Pauli form. The tensor obeys the Klein-Gordon equation

$$\left(\Box + m^2\right) T_{[\lambda\mu]\nu} = 0$$

along with the secondary conditions

$$T_\lambda = 0 \ , \quad \partial^\mu T_{[\mu\nu]\rho} = 0 = \partial^\rho T_{[\mu\nu]\rho}$$

provided the mass is non-zero.

As a consequence, in 4D spacetime the free field describes the propagation of massive modes with the five expected helicities $J_z/\hbar = \pm 2$, ± 1, and 0.

But in 4D spacetime, perhaps surprisingly, if $m = 0$ the above Lagrangian describes *no* propagating modes!

This point warrants a slight digression.

At least *psychologically*, there is a *hint* that this might be the case, based on the Euler density.

There is another way to write the field strength bilinear in the Lagrangian, namely,

$$F_{[\lambda\mu\nu]\rho}\ F^{[\lambda\mu\nu]\rho} - 3F_{[\mu\nu]}\ F^{[\mu\nu]} = \frac{3}{2}\left(R_{[\lambda\mu][\nu\rho]}R^{[\lambda\mu][\nu\rho]} - 4R_{\lambda\nu}R^{\lambda\nu} + R^2\right)$$

where $R_{[\lambda\mu][\nu\rho]}$ and its traces are given by

$$R_{[\lambda\mu][\nu\rho]} = \partial_\nu T_{[\lambda\mu]\rho} - \partial_\rho T_{[\lambda\mu]\nu}$$

$$R_{\lambda\nu} = g^{\mu\rho} R_{[\lambda\mu][\nu\rho]}\ , \quad R = g^{\lambda\nu} R_{\lambda\nu}$$

Moreover, in 4D spacetime

$$R_{[\lambda\mu][\nu\rho]}R^{[\lambda\mu][\nu\rho]} - 4R_{\lambda\nu}R^{\lambda\nu} + R^2 = -\frac{1}{4}\ \varepsilon^{\lambda\mu\gamma\delta}\ \varepsilon^{\alpha\beta\nu\rho}\ R_{[\lambda\mu][\nu\rho]}\ R_{[\alpha\beta][\gamma\delta]}$$

Now, if $R_{[\lambda\mu][\nu\rho]}$ were the Riemann curvature, this would be the Euler density — a total divergence — i.e. a topological term in the action, and therefore it would not contribute to the field equations nor to mode propagation.

But this hint is only psychological — it is actually *not* a fact. By definition, unlike the Riemann curvature,

$$R_{[\lambda\mu][\nu\rho]} \neq R_{[\nu\rho][\lambda\mu]}$$

So the field strength terms in $\mathcal{L}$ are *not* a total divergence, and therefore they *do* contribute to the field equations.

To see that no modes are propagated for the massless field in 4D requires a careful examination of all the gauge invariances of the field strength terms in $\mathcal{L}$.

The action for $\mathcal{L}$ with $m = 0$ is invariant under gauge transformations on $T_{[\lambda\mu]\nu}$ given by [2]

$$\delta T_{[\lambda\mu]\nu} = \partial_\lambda S_{\mu\nu} - \partial_\mu S_{\lambda\nu} + \partial_\lambda A_{\mu\nu} - \partial_\mu A_{\lambda\nu} + 2\partial_\nu A_{\mu\lambda}$$

where $S_{\mu\nu}$ is any local, differentiable, symmetric tensor field, and $A_{\mu\nu}$ is any local, differentiable, antisymmetric tensor field.

Now, the $m = 0$ action for $\mathcal{L}$ is actually invariant under these gauge transformations for any number of spacetime dimensions, and in general for more than 4D there will be propagating modes even for $m = 0$. However, in the special case of 4D spacetime, it so happens that these gauge transformations eliminate all propagating modes.

That is to say, in 4D the only solutions of the massless free field equations for $T_{[\lambda\mu]\nu}$ are pure gauge configurations.

Let's get back to the massive case.

The number of $T_{[\lambda\mu]\nu}$ propagating modes in 4D spacetime jumps from none, for $m = 0$, to five for $m \neq 0$. At least that is the case for the free theory. But what about coupling to other fields, or self-coupling of $T_{[\lambda\mu]\nu}$ to itself?

Some time ago [3] Peter Freund and I proposed the following.

$$\left(\Box + m^2\right) T_{[\lambda\mu]\nu} = \kappa \left(2\varepsilon_{\lambda\mu\alpha\beta}\partial^\alpha \Theta^\beta{}_\nu + \varepsilon_{\nu\mu\alpha\beta}\partial^\alpha \Theta^\beta{}_\lambda - \varepsilon_{\nu\lambda\alpha\beta}\partial^\alpha \Theta^\beta{}_\mu\right)$$

where once again $\Theta_{\mu\nu}$ is an energy-momentum tensor. Several features of this field equation warrant comments.

• First, the RHS is conserved w.r.t. all three indices if $\partial^\beta \Theta_{\alpha\beta} = 0$. Hence the divergences of $T_{[\lambda\mu]\nu}$ decouple (i.e. are free fields) and may be consistently set to zero for this interaction.

• Second, the RHS is traceless w.r.t. $\mu = \nu$ if $\Theta_{\alpha\beta} = \Theta_{\beta\alpha}$. (NB It is *not* necessary that $\Theta_{\alpha\beta}$ be traceless.) Hence the trace of $T_{[\lambda\mu]\nu}$ decouples and may also be consistently set to zero for this interaction.

Therefore the model avoids the spurious trace degree of freedom problem that plagued massive $h_{\mu\nu}$.

• Third, the RHS is a total divergence.

This means under "normal" conditions, where $\Theta_{\alpha\beta}(x)$ falls to zero sufficiently rapidly as x^μ approaches infinity on some space-like hypersurface, the source on the RHS is a *chargeless* source for most (but not all?) components of $T_{[\lambda\mu]\nu}$. (See the fourth point to follow.)

Is this related to Mach's principle? Or holography?

• Fourth, the proposed field equation implies $T_{[\lambda\mu]\nu}$ has negative parity.

How might that be reconciled with the expected positive parity of gravity?

In principle, this last question is easily answered in 4D spacetime.

It is straightforward to construct a positive parity, rank two tensor field from the field strength for $T_{[\lambda\mu]\nu}$, namely,

$$K_{\mu\nu} = \varepsilon_{\mu}^{\ \alpha\beta\gamma} F_{[\alpha\beta\gamma]\nu}$$

This illustrates a common feature in duality theory, where the field of one variable is the field strength of another.

But more importantly for the situation at hand,

$$K_{\mu}^{\ \mu} \equiv 0 \ , \quad \partial^{\mu} K_{\mu\nu} \equiv 0$$

Actually, given the field equation that Peter Freund and I proposed (i.e. "on-shell") one can show for this definition of $K_{\mu\nu}$ only the components symmetric under $\mu \leftrightarrow \nu$ couple locally to energy and momentum. The antisymmetric part decouples, i.e. $K_{\mu\nu} - K_{\nu\mu}$ is a free field.

Given the proposed field equation for $T_{[\lambda\mu]\nu}$ the on-shell equation for $K_{\mu\nu}$ is

$$\left(\Box + m^2\right) K_{\mu\nu} = 12\kappa\Box\Theta_{\mu\nu} + 4\kappa\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right)\Theta$$

Note the RHS is symmetric and manifestly traceless in 4D, as well as conserved. So $H_{\mu\nu} = K_{\mu\nu} + K_{\nu\mu}$ couples to $\Theta_{\mu\nu}$ but $K_{\mu\nu} - K_{\nu\mu}$ does not.

Also note that static energy-momentum sources *do* produce K_{00}.

$$\left(\nabla^2 - m^2\right) K_{00} = 12\kappa\nabla^2\Theta_{00} - 4\kappa\nabla^2\Theta$$

For either traceless $\Theta_{\mu\nu}$ or stress-free matter, this is similar to conventional massive gravity, but with $K_{00} \propto \nabla^2 h_{00}$.

There is a more palatable way to write the field equation for $K_{\mu\nu}$ through use of a highly *nonlinear* field redefinition.

Let

$$H_{\mu\nu} = \frac{-1}{12m^2} \left(K_{\mu\nu} - 12\kappa\Theta_{\mu\nu} \right)$$

Then

$$\left(\Box + m^2 \right) H_{\mu\nu} = \kappa\Theta_{\mu\nu} + \frac{\kappa}{3m^2} \left(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu \right) \Theta$$

This shows that it is consistent to have

$$H = H_\mu{}^\mu = \frac{\kappa}{m^2}\, \Theta$$

since $(\Box + m^2) \left(H - \frac{\kappa}{m^2}\, \Theta \right) = 0.$

More importantly, the equation for $H_{\mu\nu}$ is just the OP equation for $h_{\mu\nu}$, i.e.

$$\left(\Box + m^2\right) h_{\mu\nu} = \kappa \, \Theta_{\mu\nu} + \frac{\kappa}{3m^2} \left(\eta_{\mu\nu}\Box - \partial_\mu \partial_\nu\right) \Theta$$

So the $T_{[\lambda\mu]\nu}$ theory is the exact massive dual of the OP model, with the expected realization of one field as the field strength of the other, albeit with some nonlinear embellishments in this realization due to the interaction.

• Finally, then, what is the interaction Lagrangian that leads to the proposed field equations for $T_{[\lambda\mu]\nu}$ if $\Theta_{\alpha\beta}$ is due to $T_{[\lambda\mu]\nu}$ itself?

So far the requisite $\mathcal{L}_{int}$ is only known to first order in κ [5, 6].

It helps to first write the field equation more compactly as

$$\left(\Box + m^2\right) T_{[\lambda\mu]\nu} = \kappa P_{\lambda\mu\nu,\alpha\beta\gamma} \partial^\alpha \Theta^{\beta\gamma} \ ,$$

where we have defined a symmetrizing tensor

$$P_{\lambda\mu\nu,\alpha\beta\gamma} = 2\varepsilon_{\lambda\mu\alpha\beta}\eta_{\nu\gamma} + \varepsilon_{\nu\mu\alpha\beta}\eta_{\lambda\gamma} - \varepsilon_{\nu\lambda\alpha\beta}\eta_{\mu\gamma} \ .$$

Then the lowest order interaction is

$$\begin{aligned}
\mathcal{L}_{int} &= \frac{1}{36}\,\kappa\ K_\alpha{}^\beta K_\beta{}^\gamma K_\gamma{}^\alpha \\
&\quad + \kappa\ T_{[\lambda\mu]\nu} P^{\lambda\mu\nu,\alpha\beta}{}_\gamma\ \partial_\alpha \left(\left(\Box + m^2\right) \left(T_{[\beta\rho]\sigma} T^{[\gamma\rho]\sigma}\right) - \partial^\gamma \partial_\rho \left(T^{[\rho\sigma]\tau} T_{[\beta\sigma]\tau}\right)\right) \ ,
\end{aligned}$$

In order for this to give the sought-for field equation, the energy-momentum tensor needs an “improvement.”

$$\begin{aligned}
\Theta_\beta{}^\gamma &= \theta_\beta{}^\gamma - 36\vartheta_\beta{}^\gamma \ , \quad \partial^\beta \vartheta_\beta{}^\gamma \equiv 0 \ , \\
\vartheta_\beta{}^\gamma &\equiv \Box\left(T_{[\beta b]c} T^{[\gamma b]c}\right) - \partial_\beta \partial^a \left(T_{[ab]c} T^{[\gamma b]c}\right) - \partial^\gamma \partial_a \left(T^{[ab]c} T_{[\beta b]c}\right) + \delta_\beta{}^\gamma \partial^d \partial_a \left(T^{[ab]c} T_{[db]c}\right) \ .
\end{aligned}$$

Explicitly, in N spacetime dimensions (not just $N=4$) the on-shell free-field energy-momentum tensor is given by

$$\begin{aligned}\theta_\mu^{\ \nu} &= \mathcal{K}_\mu^{\ \lambda}\mathcal{K}_\lambda^{\ \nu} + \frac{(-1)^{N-1}\,m^2}{(N-3)!}\,T_{[\mu\alpha_2\cdots\alpha_{N-2}]\lambda}T^{[\nu\alpha_2\cdots\alpha_{N-2}]\lambda}\\ &\quad -\frac{1}{2}\,\delta_\mu^{\ \nu}\left(\mathcal{K}_{\alpha\beta}\mathcal{K}^{\beta\alpha} - \frac{(-1)^{N}\,m^2}{(N-2)!}\,T_{[\alpha_1\cdots\alpha_{N-2}]\gamma}T^{[\alpha_1\cdots\alpha_{N-2}]\gamma}\right)\ ,\end{aligned}$$

where $\mathcal{K}_\mu^{\ \nu} \equiv \frac{1}{(N-1)!}\,K_\mu^{\ \nu}$, and the improved tensor is given by

$$\begin{aligned}\Theta_\mu^{\ \nu} &= \theta_\mu^{\ \nu} + \frac{(-1)^{N-1}}{(N-3)!}\,\vartheta_\mu^{\ \nu}\ ,\\ \vartheta_\mu^{\ \nu} &\equiv \Box\left(T_{[\mu\alpha_2\cdots\alpha_{N-2}]c}T^{[\nu\alpha_2\cdots\alpha_{N-2}]c}\right) + \delta_\mu^{\ \nu}\partial_a\partial^b\left(T^{[a\alpha_2\cdots\alpha_{N-2}]c}T_{[b\alpha_2\cdots\alpha_{N-2}]c}\right)\\ &\quad -\partial_\mu\partial^b\left(T_{[b\alpha_2\cdots\alpha_{N-2}]c}T^{[\nu\alpha_2\cdots\alpha_{N-2}]c}\right) - \partial^\nu\partial_b\left(T^{[b\alpha_2\cdots\alpha_{N-2}]c}T_{[\mu\alpha_2\cdots\alpha_{N-2}]c}\right)\ .\end{aligned}$$

But what is $\mathcal{L}_{int}$ to all orders in κ?

I do not yet know the answer to this last question. But I *suspect* it will involve

$$\det\left(1+\kappa K_{\alpha}{}^{\beta}\right)=1-\frac{1}{2}\,\kappa^{2}K_{\lambda}{}^{\mu}K_{\mu}{}^{\lambda}+\frac{1}{3}\,\kappa^{3}K_{\lambda}{}^{\mu}K_{\mu}{}^{\nu}K_{\nu}{}^{\lambda}+\frac{1}{8}\,\kappa^{4}\left(\left(K_{\lambda}{}^{\mu}K_{\mu}{}^{\lambda}\right)^{2}-2K_{\lambda}{}^{\mu}K_{\mu}{}^{\nu}K_{\nu}{}^{\rho}K_{\rho}{}^{\lambda}\right)$$

Note the κ^2 term here is a very compact way to express the field strength bilinear in the $T_{[\lambda\mu]\nu}$ Lagrangian.

$$\frac{1}{6}\,K_{\lambda}{}^{\mu}K_{\mu}{}^{\lambda}=-F_{[\lambda\mu\nu]\rho}\,F^{[\lambda\mu\nu]\rho}+3F_{[\mu\nu]}\,F^{[\mu\nu]}$$

Under a gauge transformation $S_{\mu\nu}$ gauge parameters drop out of $\delta F_{[\lambda\mu\nu]\rho}$, so

$$\delta K_{\mu\nu}=-2\partial_{\nu}V_{\mu}\ ,\quad V_{\mu}\equiv\varepsilon_{\mu\alpha\beta\gamma}\left(\partial^{\alpha}A^{\beta\gamma}+\partial^{\beta}A^{\gamma\alpha}+\partial^{\gamma}A^{\alpha\beta}\right)$$

Invariance of the free field action then follows immediately upon integration by parts, discarding any surface contributions.

$$\delta\int K_{\mu}{}^{\nu}K_{\nu}{}^{\mu}d^{4}x=-4\int\left(\partial^{\nu}V_{\mu}\right)K_{\nu}{}^{\mu}d^{4}x\underset{\text{i.b.p.}}{=}4\int V_{\mu}\left(\partial^{\nu}K_{\nu}{}^{\mu}\right)d^{4}x=0$$

In any case, with my student Hassan Alshal, I have made some progress towards the answer to this last question, and to similar problems in higher dimensional spacetimes. In particular, we have developed the theory for a scalar analogue of the $T_{[\lambda\mu]\nu}$ model.

The analogous scalar problem

In 4D a massive scalar Φ is dual to a totally antisymmetric tensor field

$$V_{\alpha\beta\gamma} = \varepsilon_{\alpha\beta\gamma\mu}\partial^{\mu}\Phi$$

whose totally antisymmetric gauge invariant field strength is

$$F_{\lambda\mu\nu\rho} = \partial_{\lambda}V_{\mu\nu\rho} - \partial_{\mu}V_{\nu\rho\lambda} + \partial_{\nu}V_{\rho\lambda\mu} - \partial_{\rho}V_{\lambda\mu\nu}$$

and whose free field Lagrangian is

$$\mathcal{L} = -\frac{1}{48}\, F_{\lambda\mu\nu\rho}\, F^{\lambda\mu\nu\rho} + \frac{1}{12}\, m^2 V_{\lambda\mu\nu} V^{\lambda\mu\nu}$$

As was the case for $T_{[\lambda\mu]\nu}$, when $m = 0$ the free field theory governed by $\mathcal{L}$ has no propagating modes, but when $m \neq 0$ the number of modes jumps up to one in this case — corresponding to Φ.

The model analogous to the $T_{[\lambda\mu]\nu}$ theory proposed above has field equations

$$\begin{aligned}\left(\Box + m^2\right) V_{\alpha\beta\gamma} &= \kappa\varepsilon_{\alpha\beta\gamma\mu}\partial^\mu\Theta \\ \partial^\alpha V_{\alpha\beta\gamma} &= 0\end{aligned}$$

where $\Theta = \Theta_\nu{}^\nu$ is the trace of an energy-momentum tensor.

Peter Freund and I also proposed this model some time ago [3], but we did not exhibit a Lagrangian which led to this theory for a self-coupled $V_{\alpha\beta\gamma}$, i.e. when $\Theta_{\alpha\beta}$ is an energy-momentum tensor for $V_{\alpha\beta\gamma}$ itself.

When I was invited to contribute to a volume honoring Peter, who died in March 2018, I decided to revisit this problem and find an $\mathcal{L}$ for the self-coupled theory. I succeeded [4].

The result is

$$\mathcal{L} = -\frac{1}{2}\, m^2 u + \frac{1}{2}\left(v - m^2\kappa u\right)^2 + \frac{1}{\kappa^2}\, F\left(\kappa\left(v - m^2\kappa u\right)\right)$$

$$u \equiv -\frac{1}{6}\, V_{\lambda\mu\nu} V^{\lambda\mu\nu}\ , \quad v \equiv \frac{1}{24}\, \varepsilon^{\lambda\mu\nu\rho} F_{\lambda\mu\nu\rho}$$

where

$$\begin{aligned} F\left(w\right) &= -\frac{1}{2}\, w + \frac{1}{4}\, w\sqrt{1+4w^2} + \frac{1}{8}\, \ln\left(2w + \sqrt{1+4w^2}\right) \\ &\equiv \frac{1}{3}\, w^3\, {}_3F_2\left(1, \frac{1}{2}, \frac{3}{2}; 2, \frac{5}{2}; -4w^2\right) \\ &= \frac{1}{3}\, w^3 - \frac{1}{5}\, w^5 + \frac{2}{7}\, w^7 - \frac{5}{9}\, w^9 + \frac{14}{11}\, w^{11} - \frac{42}{13}\, w^{13} + \frac{44}{5}\, w^{15} + O\left(w^{17}\right)\ . \end{aligned}$$

My student and I generalized this result [6] to any number of spacetime dimensions, N. In that case a massive scalar is dual to a totally antisymmetric tensor of rank $N-1$,

$$V_{\alpha_1\cdots\alpha_{N-1}} = \varepsilon_{\alpha_1\cdots\alpha_N}\partial^{\alpha_N}\Phi$$

with a gauge invariant field strength that is totally antisymmetric of rank N.

The field equations are now

$$\begin{aligned} \left(\Box + m^2\right) V_{\alpha_1\cdots\alpha_{N-1}} &= \kappa\varepsilon_{\alpha_1\cdots\alpha_N}\partial^{\alpha_N}\Theta \\ \partial^{\alpha_1} V_{\alpha_1\cdots\alpha_{N-1}} &= 0 \end{aligned}$$

Again there is a closed-form expression for the Lagrangian that gives this interacting theory when Θ depends on the field $V_{\alpha_1\cdots\alpha_{N-1}}$ itself.

To obtain $\mathcal{L}$ in this case, it is necessary to find the root of the trinomial equation

$$X^{\frac{N}{N-2}} = 1 + zX$$

that goes to 1 as $z \to 0$. That is to say

$$X^N - (1 + zX)^{N-2} = 0$$

At first glance this looks very difficult for $N \neq 4$, but in fact there is a closed-form expression for the root [7, 8]. As a series it is

$$X(z) = \frac{N-2}{N} \sum_{m=0}^{\infty} \frac{z^m}{m!} \frac{\Gamma\left(\frac{N-2}{N}(1+m)\right)}{\Gamma\left(2 - \frac{2}{N}(1+m)\right)}$$

This series is a special case of a generalization for the confluent hypergeometric function ${}_1F_1$, as defined by [9]

$${}_1\mathcal{F}_1\left(\begin{matrix}\alpha \\ \beta\end{matrix}; \begin{matrix}\rho \\ \sigma\end{matrix}; z\right) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \frac{\Gamma(\alpha + \beta m)}{\Gamma(\rho + \sigma m)}$$

In due course, we hope to find results of this type for the $T_{[\lambda\mu]\nu}$ field, and its generalization to N-dimensional spacetimes. If you like to shuffle indices, perhaps this is your calling.

Acknowledgement: Thank you for your time, and for the opportunity to visit Kansas and give this talk.

References

[1] In a paper, it is considered "bad taste" to cite wikipedia rather than the original research articles. But this is a talk, not a paper, so ... https://en.wikipedia.org/wiki/Massive_gravity. Many of the original papers on massive gravity are referenced in this wiki article, including Fierz and Pauli. But not mine! See [2, 3] and more recently [4, 5, 6].

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