

Conducting Ellipsoids in Uniform Motion

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Abstract

This is a survey of the electromagnetic fields produced by a charged conducting ellipsoid, either when static or else when moving uniformly at a constant velocity in a given inertial frame. Results are presented for any number of spatial dimensions.

1 Introduction

The electrostatics of charged conducting ellipsoids were first understood in the early part of the nineteenth century [1]. The surface charge densities as well as the potentials and electric fields surrounding such objects have elegant geometrical properties, as discussed extensively in the literature [2, 3, 4, 5, 6]. In particular, the equipotentials are confocal ellipsoids surrounding the charged surface, with the electric field everywhere normal to those equipotentials. More specifically, the electric field at any observation point outside a charged conducting ellipsoid of revolution, either prolate or oblate, is always directed along the bisector of a pair of straight lines drawn from either of two focal points of the ellipsoid to the observation point. Moreover, for any conducting ellipsoid in three spatial dimensions, when projected along any of the three principal axes the charge per length is always constant, a feature that is perhaps the one elementary property that is easiest to keep in mind. However, in any other number of spatial dimensions, this last statement must be modified, as discussed recently in [7].

In any case, it may *not* be so well-known that all of these geometric properties have simple analogues when the ellipsoid is set in motion at a constant velocity.² The purpose of this article is to discuss various geometric features of the electromagnetic fields for uniformly moving conducting ellipsoids, in any number of spatial dimensions.

2 Electrostatic ellipsoids

2.1 Charge densities

Let us begin with the charge distribution on a stationary conducting ellipsoidal surface embedded in three spatial dimensions. The charged surface is specified by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \tag{1}$$

where we have expressed the constraint that defines the ellipsoid in the most convenient Cartesian frame. If this ideal surface is a conductor carrying a total charge Q , i.e. an equipotential surface, then that charge

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²We mostly consider cases where the velocity of the moving ellipsoid is directed along one of the principal axes, for simplicity.

is distributed according to a *volume* charge density given simply by

$$\rho(\vec{r}) = \frac{Q}{4\pi abc} \delta\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right) \quad (2)$$

The Dirac delta $\delta\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right)$ restricts all the charge to lie on the surface, albeit not uniformly if any two of a, b, c are unequal. Of course, when $a = b = c$ the expression (2) reduces to $\rho(\vec{r})$ for a uniformly charged sphere of radius a . For a complete justification of (2), we next compute the corresponding surface charge density on the general a, b, c ellipsoid. The point is just that, for a volume density of the form $\rho(\vec{r}) = f(\vec{r}) \delta\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right)$, the restriction of the function $f(\vec{r})$ to the surface is uniquely determined by the surface charge density.

To obtain the non-uniform surface charge density for arbitrary a, b, c it is necessary to integrate $\rho(\vec{r})$ along a line normal to the charged surface. This calculation goes as follows. The normal unit vector at any point *on the surface* is given by

$$\hat{n} = \frac{\vec{\nabla}(x^2/a^2 + y^2/b^2 + z^2/c^2)}{\left|\vec{\nabla}(x^2/a^2 + y^2/b^2 + z^2/c^2)\right|} = \frac{x \hat{x}/a^2 + y \hat{y}/b^2 + z \hat{z}/c^2}{\sqrt{x^2/a^4 + y^2/b^4 + z^2/c^4}} \quad (3)$$

while the three-space volume element in an infinitesimal neighborhood straddling the surface is

$$dV = du dA, \quad du = \hat{n} \cdot d\vec{r} \quad (4)$$

Rewriting the Dirac delta in terms of the normal coordinate u then gives

$$\begin{aligned} \delta\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right) &= \frac{1}{\left|\frac{d}{du}\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}\right|} \delta(u - u_0) = \frac{1}{\left|\hat{n} \cdot \vec{\nabla}\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}\right|} \delta(u - u_0) \\ &= \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} \delta(u - u_0) \end{aligned} \quad (5)$$

where³ $u = u_0$ when $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Using this last expression and integrating over the infinitesimal neighborhood $u \in (u_0 - \varepsilon, u_0 + \varepsilon)$ gives the expected result (e.g. see [5]) for the surface charge density,

$$\sigma(\vec{r}) = \lim_{\varepsilon \rightarrow 0} \int_{u_0 - \varepsilon}^{u_0 + \varepsilon} \rho(\vec{r}) du = \frac{Q}{4\pi abc} \frac{1}{\sqrt{\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}}} \quad (6)$$

thereby confirming that (2) is correct. In this last expression, it is to be understood that all points \vec{r} are on the surface (1).

The volume charge density (2) is also convenient to show that the projected charge/length along any principal axis is constant. For example, using the Dirac delta property $\delta(f(z)) = \sum_{\text{roots } z_0 \text{ of } f} \delta(z - z_0) / |f'(z_0)|$, we have for any x between $\pm a$,

$$\begin{aligned} \frac{dQ}{dx} &\equiv \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \rho(\vec{r}) = \frac{Q}{4\pi abc} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \delta\left(\sqrt{\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}} - 1\right) \\ &= \frac{Q}{4\pi abc} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dz \left| \frac{c^2}{z} \right| \left[\delta\left(z - c\sqrt{1 - x^2/a^2 - y^2/b^2}\right) + \delta\left(z + c\sqrt{1 - x^2/a^2 - y^2/b^2}\right) \right]_{u=u_0} \\ &= \frac{Q}{2\pi ab} \int_{-b\sqrt{1-x^2/a^2}}^{+b\sqrt{1-x^2/a^2}} \frac{dy}{\sqrt{1 - x^2/a^2 - y^2/b^2}} = \frac{Q}{2\pi a} \int_{-1}^{+1} \frac{ds}{\sqrt{1 - s^2}} \end{aligned} \quad (7)$$

³For a given point on the ellipsoid, the value for u_0 is unique, obviously, since the surface does not intersect itself.

But then $\int_{-1}^{+1} \frac{ds}{\sqrt{1-s^2}} = \pi$, so for $-a \leq x \leq a$,

$$\frac{dQ}{dx} = \frac{Q}{2a} \quad (8)$$

Similarly, the projected linear charge densities along the other principal axes are given by $dQ/dy = Q/(2b)$ for $-b \leq y \leq b$ and $dQ/dz = Q/(2c)$ for $-c \leq z \leq c$.

The results and analysis given above are easily extended to describe equipotential static conducting ellipsoids in any number of spatial dimensions, D . At the risk of being somewhat repetitive, let us consider this generalization in detail.

The corresponding volume charge distribution, $\rho_D(\vec{r})$, that is appropriate for equipotential ellipsoids in D dimensions, is given by an immediate generalization of (2), namely,¹

$$\rho_D(\vec{r}) = \frac{Q}{\Omega_D a_1 a_2 \cdots a_D} \delta\left(\sqrt{\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_D^2}{a_D^2}} - 1\right) = \sigma_D(\vec{r}) \delta(u - u_0) \quad (9)$$

$$\sigma_D(\vec{r}) = \frac{Q}{\Omega_D a_1 a_2 \cdots a_D} \frac{1}{\sqrt{x_1^2/a_1^4 + x_2^2/a_2^4 + \cdots + x_D^2/a_D^4}} \Big|_{u=u_0} \quad (10)$$

where $u = u_0$ when $x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_D^2/a_D^2 = 1$ and where

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)} \quad (11)$$

This Ω_D is the total ‘‘solid angle’’ in D spatial dimensions. The unit normal vector and the volume measure, for an infinitesimal neighborhood straddling the $D - 1$ dimensional ‘‘hypersurface’’ containing the charge, are also given by the obvious generalizations of (3) and (4).

$$\hat{n} = \frac{x_1 \hat{x}_1/a_1^2 + x_2 \hat{x}_2/a_2^2 + \cdots + x_D \hat{x}_D/a_D^2}{\sqrt{x_1^2/a_1^4 + x_2^2/a_2^4 + \cdots + x_D^2/a_D^4}}, \quad d^D V = du d^{D-1} V, \quad du = \hat{n} \cdot d\vec{r} \quad (12)$$

The volume charge density (9) can be projected along any principal axis to obtain a *non*-uniform linear charge density for $D \neq 3$. As done before for $D = 3$, the projection is defined by integrating over all but one direction. For instance,

$$\frac{dQ}{dx_1} \equiv \int_{-\infty}^{+\infty} dx_2 \cdots \int_{-\infty}^{+\infty} dx_D \rho_D(\vec{r}) = \frac{Q}{\Omega_D a_1 a_2 \cdots a_D} \int_{-\infty}^{+\infty} dx_2 \cdots \int_{-\infty}^{+\infty} dx_D \delta\left(\sqrt{\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_D^2}{a_D^2}} - 1\right) \quad (13)$$

Now perform the integrations sequentially, beginning with $\int_{-\infty}^{+\infty} dx_D$ using the same property of the Dirac delta as was used for $D = 3$.

$$\delta\left(\sqrt{\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \cdots + \frac{x_D^2}{a_D^2}} - 1\right) = \left|\frac{a_D^2}{x_D}\right| \left[\begin{array}{l} \delta\left(x_D - a_D \sqrt{1 - x_1^2/a_1^2 - \cdots - x_{D-1}^2/a_{D-1}^2}\right) \\ + \delta\left(x_D + a_D \sqrt{1 - x_1^2/a_1^2 - \cdots - x_{D-1}^2/a_{D-1}^2}\right) \end{array} \right]_{u=u_0} \quad (14)$$

This leads to the next integration,

$$\frac{2}{a_{D-1}} \int_{-a_{D-1} \sqrt{1 - x_1^2/a_1^2 - \cdots - x_{D-2}^2/a_{D-2}^2}}^{+a_{D-1} \sqrt{1 - x_1^2/a_1^2 - \cdots - x_{D-2}^2/a_{D-2}^2}} dx_{D-1} \frac{1}{\sqrt{1 - x_1^2/a_1^2 - \cdots - x_{D-1}^2/a_{D-1}^2}} = \frac{2\pi}{a_{D-1}} \quad (15)$$

¹It is interesting to compare the hypersurface charge density σ_D with the product of the principal curvatures, κ_m , $m = 1, 2, \dots, D - 1$, for the same ellipsoidal hypersurface. A straightforward calculation (see the Appendix) gives

$$(\Omega_D \sigma_D)^{D+1} = \left(\prod_{j=1}^D \frac{1}{a_j^{D-1}}\right) \left(\prod_{m=1}^{D-1} \kappa_m\right)$$

and therefore

$$\frac{dQ}{dx_1} = \frac{Q}{\Omega_D a_1 a_2 \cdots a_{D-2}} V(a_1, a_2, a_3, \cdots, a_{D-2}) \quad (16)$$

The remaining integrations, easily performed sequentially, are given for $-a_1 \leq x_1 \leq a_1$ by

$$\begin{aligned} V(a_1, \cdots, a_{D-2}) &= 2\pi \int_{-a_2 \sqrt{1-x_1^2/a_1^2}}^{+a_2 \sqrt{1-x_1^2/a_1^2}} dx_2 \int_{-a_3 \sqrt{1-x_1^2/a_1^2-x_2^2/a_2^2}}^{+a_3 \sqrt{1-x_1^2/a_1^2-x_2^2/a_2^2}} dx_3 \cdots \int_{-a_{D-2} \sqrt{1-x_1^2/a_1^2-\cdots-x_{D-3}^2/a_{D-3}^2}}^{+a_{D-2} \sqrt{1-x_1^2/a_1^2-\cdots-x_{D-3}^2/a_{D-3}^2}} dx_{D-2} \\ &= \Omega_{D-1} a_2 a_3 \cdots a_{D-2} (1-x_1^2/a_1^2)^{\frac{D-3}{2}} \end{aligned} \quad (17)$$

Note that $\Omega_{D-1} = \frac{2\pi}{D-3} \Omega_{D-3}$. Thus the final linear charge density projected along the axis is

$$\frac{dQ}{dx_1} = \frac{Q}{a_1} \frac{\Omega_{D-1}}{\Omega_D} (1-x_1^2/a_1^2)^{\frac{D-3}{2}} \quad (18)$$

for $-a_1 \leq x_1 \leq a_1$, in agreement with the results in §6 of [7]. Clearly, the same form applies for any other principal axis, i.e. $\frac{dQ}{dx_k} = \frac{Q}{a_k} \frac{\Omega_{D-1}}{\Omega_D} (1-x_k^2/a_k^2)^{\frac{D-3}{2}}$, for $-a_k \leq x_k \leq a_k$ and $k = 1, \cdots, D$.

For $D > 3$ the non-uniform linear charge density (18) is rather counter-intuitive as it has a maximum at $x_1 = 0$ and falls monotonically to zero on either side of the maximum, vanishing at the end points $x_1 = \pm a_1$. Only for $D = 2$ does the result conform to what one would naively expect for mutually repelling charges placed on a line, namely, a charge distribution peaked at the ends. This is discussed in [7].

As a generalization of (18), consider projecting the charge onto any subset of the principal axes, for example, onto x_1, x_2, \cdots, x_k for $k < D$, as may be accomplished by integrating ρ_D over x_m for $m = k+1, \cdots, D$. Following the same steps as given above, the result is readily seen to be

$$\frac{dQ}{dx_1 dx_2 \cdots dx_k} = \frac{Q}{a_1 a_2 \cdots a_k} \frac{\Omega_{D-k}}{\Omega_D} (1-x_1^2/a_1^2 - x_2^2/a_2^2 - \cdots - x_k^2/a_k^2)^{\frac{D-k-2}{2}} \quad (19)$$

for all x_1, x_2, \cdots, x_k such that $x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_k^2/a_k^2 \leq 1$. Since the final answer here is independent of a_{k+1}, \cdots, a_D , this would in fact be the correct charge density on an equipotential k -dimensional manifold obtained from the original equipotential ellipsoid by letting $a_m \rightarrow 0$ for $m = k+1, \cdots, D$, i.e. by ‘‘squashing’’ these $D-k$ dimensions. In particular, if the original $x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_D^2/a_D^2 = 1$ ellipsoid were completely ‘‘flattened’’ to a two-dimensional ellipse embedded in D spatial dimensions, the surface charge density that results would be

$$\frac{dQ}{dx_1 dx_2} = \frac{Q}{a_1 a_2} \frac{\Omega_{D-2}}{\Omega_D} (1-x_1^2/a_1^2 - x_2^2/a_2^2)^{\frac{D-4}{2}} \quad (20)$$

for all x_1, x_2 such that $x_1^2/a_1^2 + x_2^2/a_2^2 \leq 1$. Note that this counts the total charge on *both* sides of the final flattened ellipse, and it nicely generalizes the well-known charge per area on an ideal, flat, elliptical, equipotential disk embedded in three dimensions (for example, see [5]), namely,

$$\frac{dQ}{dA} = \frac{Q}{2\pi a_1 a_2} \frac{1}{\sqrt{1-x_1^2/a_1^2 - x_2^2/a_2^2}} \quad \text{for } D = 3. \quad (21)$$

For both (20) and (21), the boundary of the flattened disk is the ellipse $x_1^2/a_1^2 + x_2^2/a_2^2 = 1$.

As was the case for the linear charge density in (18), the result (19) is rather counter-intuitive for squashed manifolds with $D \geq k+2$. For $D < k+2$, the charge density $dQ/dx_1 dx_2 \cdots dx_k$ is peaked at the boundary of the squashed manifold, for which $x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_k^2/a_k^2 = 1$, where the density actually diverges. In our opinion, this would conform with naive expectations for mutually repelling charges placed on the manifold. But for the case $D = k+2$, the charge density on the squashed manifold is uniform, exactly like that of an ideal conducting line segment in three dimensions. Or, as another particular case, a flat, elliptic, equipotential disk in four spatial dimensions would have a constant surface charge density. So (19) for $D = k+2$ is again non-intuitive, although perhaps it can be reconciled with intuition using arguments similar to those advanced for the equipotential line segment embedded in three dimensions, as discussed in [8] and references therein. On the other hand, for $D > k+2$, the charge distribution (19) is peaked at the center of the squashed manifold and falls monotonically to zero at the boundary where $x_1^2/a_1^2 + x_2^2/a_2^2 + \cdots + x_k^2/a_k^2 = 1$, exactly like ideal line segments for $D > 3$. In our opinion, this is counter-intuitive. Nevertheless, it is what it is.

2.2 Static potentials and electric fields

An exact, single-parameter integral expression for the potential surrounding a static, equipotential, conducting ellipsoid carrying a total charge Q , in D spatial dimensions, is in general an elliptic integral, although it may reduce to an elementary function if some of the a_k are equal. Explicitly, if the charged surface is defined by

$$\sum_{k=1}^D \frac{x_k^2}{a_k^2} = 1 \quad (22)$$

then the potential is given by

$$\Phi(\vec{r}) = \frac{kQ}{2} \int_{\Theta(\vec{r})}^{\infty} \left(\prod_{k=1}^D \frac{1}{\sqrt{a_k^2 + \theta}} \right) d\theta \quad (23)$$

where the Θ -equipotentials are a set of *confocal* ellipsoids consisting of all points \vec{r} outside the charged ellipsoid that satisfy, for a given Θ ,

$$\sum_{k=1}^D \frac{x_k^2}{a_k^2 + \Theta} = 1, \quad \text{for } \Theta > 0 \quad (24)$$

Note the charged ellipsoid itself is defined to be at $\Theta = 0$. If given an arbitrary point \vec{r} outside the charged ellipsoid, to compute the potential at that point it would first be necessary to find the appropriate Θ for the given \vec{r} , i.e. find $\Theta(\vec{r})$. In general, if all the a_k are distinct, this would require solving for the appropriate root of the D th order polynomial in Θ implicit in (24), something that can always be done in principle (although in practice, perhaps only numerically, especially if $D > 4$ and all the a_k are distinct).

The static electric field is the gradient of the potential, as usual. From (24) and (23) it follows that

$$\vec{\nabla}_{\Theta}(\vec{r}) = \left(\sum_{n=1}^D \frac{2x_n \widehat{e}_n}{a_n^2 + \Theta} \right) / \left(\sum_{m=1}^D \frac{x_m^2}{(a_m^2 + \Theta)^2} \right) \quad (25)$$

$$\vec{E}(\vec{r}) = -\vec{\nabla}\Phi = -(\vec{\nabla}\Theta) \frac{d\Phi}{d\Theta} = kQ \left(\prod_{k=1}^D \frac{1}{\sqrt{a_k^2 + \Theta}} \right) \left(\sum_{n=1}^D \frac{x_n \widehat{e}_n}{a_n^2 + \Theta} \right) / \left(\sum_{m=1}^D \frac{x_m^2}{(a_m^2 + \Theta)^2} \right) \quad (26)$$

Once again, if only \vec{r} is specified, it is necessary to find $\Theta(\vec{r})$ from (24) to evaluate this expression. But note that as $r \rightarrow \infty$, it follows from (24) that $\Theta \rightarrow \infty$ with $\Theta \underset{r \rightarrow \infty}{\sim} r^2$. So asymptotically,

$$\Phi(\vec{r}) \underset{r \rightarrow \infty}{\sim} \frac{kQ}{D-2} \frac{1}{r^{D-2}}, \quad \vec{E} \underset{r \rightarrow \infty}{\sim} \frac{kQ}{r^D} \vec{r} \quad (27)$$

These asymptotic expressions are just the potential and electric field for a point charge in D spatial dimensions (e.g. see [4]) as should have been expected.

The direction of the electric field (when multiplied by the sign of Q) at a point on a given Θ -equipotential ellipsoid, is given by the unit vector

$$\widehat{E}(\vec{r}) = \left(\sum_{n=1}^D \frac{x_n \widehat{e}_n}{a_n^2 + \Theta} \right) / \sqrt{\sum_{m=1}^D \frac{x_m^2}{(a_m^2 + \Theta)^2}} \quad (28)$$

As a check, when all the a_k are equal, $\widehat{E} = \widehat{r}$, as expected. The (signed) strength of the electric field, at a point on that same equipotential, is given by

$$E(\vec{r}) = kQ \left(\prod_{k=1}^D \frac{1}{\sqrt{a_k^2 + \Theta}} \right) / \sqrt{\sum_{m=1}^D \frac{x_m^2}{(a_m^2 + \Theta)^2}} \quad (29)$$

so that $\vec{E}(\vec{r}) = E(\vec{r}) \hat{E}(\vec{r})$ for either sign of Q . This reduces to $E = kQ/r^{D-1}$ when all the a_k are equal, as expected.

Since the charge is located on the ellipsoid with $\Theta = 0$, by construction, the potential on the charge carrying hypersurface itself is

$$\Phi(\vec{r}(\Theta = 0)) = \frac{kQ}{2} \int_0^\infty \left(\prod_{k=1}^D \frac{1}{\sqrt{a_k^2 + \theta}} \right) d\theta \quad (30)$$

That is to say, the capacitance of the isolated ellipsoid, defined by $Q = C\Phi(\vec{r}(\Theta = 0))$, is also given by an elliptic integral

$$C = \frac{2}{k \int_0^\infty \left(\prod_{k=1}^D \frac{1}{\sqrt{a_k^2 + \theta}} \right) d\theta} \quad (31)$$

Furthermore, the charge density on the ellipsoidal hypersurface may be obtained directly from Gauss' law, with the normalization for a point charge determined by

$$\vec{\nabla} \cdot \frac{\vec{r}}{r^D} = \Omega_D \delta^D(\vec{r}) \quad (32)$$

where again Ω_D is the total solid angle in D spatial dimensions. Thus the hypersurface charge density is given as usual by the value of the normal electric field as the hypersurface is approached from the outside, that is to say by the limit: $k \Omega_D \sigma_D(\vec{r}) = \lim_{\vec{r} \rightarrow \text{hypersurface}} \hat{n} \cdot \vec{E}$ where \hat{n} is the outward normal unit vector on the hypersurface. For the problem at hand this limit is just $\lim_{\Theta \rightarrow 0} \hat{E} \cdot \vec{E} = E|_{\Theta=0}$. So (29) gives the explicit result

$$\sigma_D(\vec{r}) = \frac{1}{k \Omega_D} E|_{\Theta=0} = \frac{Q}{\Omega_D \prod_{k=1}^D a_k} \frac{1}{\sqrt{\sum_{m=1}^D \frac{x_m^2}{a_m^4}}} \Big|_{\Theta=0} \quad (33)$$

thereby confirming both (9) and (10).⁴

3 Moving ellipsoids

To obtain electromagnetic fields for an ellipsoidal conductor moving uniformly, it is only necessary to Lorentz boost the static conducting surface defined in its "rest frame" by (22).

Consider a Lorentz boost from the inertial frame of observer O to another inertial frame, of observer O' , where the coordinate origin of O' moves along a constant velocity trajectory given by $\vec{r}' = \vec{v}t$ in the frame of observer O . We suppose the two frames have the same spatial orientation, with unit vectors $\hat{e}_k = \hat{e}'_k$ for $k = 1, \dots, D$. A spacetime point (ct, \vec{r}) in the O frame is then given in the O' frame by

$$ct' = \gamma(ct - \vec{v} \cdot \vec{r}/c), \quad \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \quad (34)$$

$$\vec{r}' = (\vec{r} - \hat{v} \hat{v} \cdot \vec{r}) + \gamma(\hat{v} \cdot \vec{r} - vt)\hat{v}, \quad \vec{v} \equiv v \hat{v} \quad (35)$$

These expressions are valid for any number of spatial dimensions. However, for $D = 3$, the spatial part of \vec{r}' orthogonal to \vec{v} may be expressed in a familiar form using cross-products, namely, $\vec{r}' - \hat{v} \hat{v} \cdot \vec{r}' = -\hat{v} \times (\hat{v} \times \vec{r})$.

An ellipsoid written in the canonical form (22), but at rest with respect to observer O' , is therefore seen by O as a moving surface given by

$$1 = \sum_{k=1}^D \frac{(x'_k)^2}{a_k^2} = \sum_{k=1}^D \frac{1}{a_k^2} \{x_k + v_k [(\gamma - 1)(\vec{v} \cdot \vec{r}) - \gamma v^2 t] / v^2\}^2 \quad (36)$$

⁴Using notation that is more consistent with the previous subsection, $d\Theta = (\vec{\nabla}\Theta) \cdot d\vec{r} = |\vec{\nabla}\Theta| \hat{n}(\Theta) \cdot d\vec{r} = |\vec{\nabla}\Theta| du(\Theta)$, where $\hat{n}(\Theta) = \hat{E}$ is the local normal on the Θ -equipotential, and $|\vec{\nabla}\Theta| = 2 / \sqrt{\sum_{m=1}^D x_m^2 / (a_m^2 + \Theta)^2}$. Therefore $\vec{E} = -\hat{n}(\Theta) d\Phi/du(\Theta) = -\hat{E} |\vec{\nabla}\Theta| d\Phi/d\Theta$. This is in agreement with (26), (28), and (29).

upon using $x'_k = \vec{r}' \cdot \hat{e}'_k = \vec{r}' \cdot \hat{e}_k = x_k + v_k [(\gamma - 1)(\vec{v} \cdot \vec{r}') - \gamma v^2 t] / v^2$. In general, this is still an ellipsoid, centered on the point $\vec{v}t$ (i.e. the origin of O'), but if \vec{v} is *not* directed along any of the principal axes of the original ellipsoid, the moving surface is *not* an ellipsoid in canonical form due to the presence of the terms $x_k x_m$ for $k \neq m$ in (36). To express the moving surface in canonical form for observer O , it would be necessary to perform a spatial reorientation (i.e. a rotation) of the frame for O . We will **not** consider this general case in the following. Rather, we will only consider situations where the boost is along a principal axis of the original ellipsoid.

3.1 Motion along a principal axis

If the boost is to a new inertial frame moving along the D th axis, with both inertial frames coincident at $t = 0$, then in the frame where the center of the ellipsoid has velocity $\vec{v} = v \hat{e}_D$ and position $\vec{r} = \vec{v}t$, passing through the origin at $t = 0$, the moving charged surface is defined by the Lorentz transformation of (22) so that points \vec{r} on the charged surface now obey, at any given time t ,

$$\sum_{k=1}^{D-1} \frac{x_k^2}{a_k^2} + \frac{\gamma^2 (x_D - vt)^2}{a_D^2} = 1 \quad (37)$$

This is again an ellipsoid, but of course it has been Lorentz contracted along the direction of motion.

Moreover, the equipotential Θ -ellipsoids are also Lorentz contracted along the direction of motion to become moving surfaces now defined by “ (Θ, v) -ellipsoids”:

$$\sum_{k=1}^{D-1} \frac{x_k^2}{a_k^2 + \Theta} + \frac{\gamma^2 (x_D - vt)^2}{a_D^2 + \Theta} = 1, \quad \text{for } \Theta > 0, \quad (38)$$

Each of these moving (Θ, v) -ellipsoids is centered on the point $(0, \dots, 0, vt)$. However, at any fixed t , for different values of Θ these moving ellipsoids are *no longer confocal* for planes containing the D axis just because the last term on the LHS of (38) is now $\frac{(x_D - vt)^2}{a_D^2 / \gamma^2 + \Theta / \gamma^2}$ and therefore has an effective Θ that is less than that for the other axes. The moving charged surface itself is again given by $\Theta = 0$, as it was for the static case. For any given point \vec{r} outside the charged surface defined by (37), for any given t and v , the equation (38) implicitly defines the value of Θ for the moving ellipsoidal surface that instantaneously includes the given point.

Since the previous volume charge density transforms as a component of the Lorentz vector $J^\mu = (c\rho, \vec{J})$, where the charged static ellipsoid has $\vec{J} = 0$, the charge and current densities for the moving D -dimensional ellipsoid are now just

$$\rho_D(\vec{r}, t) = \frac{Q}{\Omega_D a_1 a_2 \dots a_D} \delta \left(\sqrt{\sum_{k=1}^{D-1} \frac{x_k^2}{a_k^2} + \frac{\gamma^2 (x_D - vt)^2}{a_D^2}} - 1 \right) \quad (39)$$

$$\vec{J}(\vec{r}, t) = \vec{v} \rho_D(\vec{r}, t) \quad (40)$$

Also, on any particular (Θ, v) -ellipsoid defined by (38) the values of the scalar and vector potentials are also given by a boost of the Lorentz vector $(\Phi/c, \vec{A})$.⁵ Thus

$$\Phi(\vec{r}, t) = \gamma \Phi_{\text{rest}}(\Theta) \quad (41)$$

$$\vec{A}(\vec{r}, t) = \vec{v} \gamma \Phi_{\text{rest}}(\Theta) / c^2 \quad (42)$$

where the “comoving” or “rest frame” potential $\Phi_{\text{rest}}(\Theta)$ is given by exactly the same expression as for the previous static case, namely, by the elliptic integral:

$$\Phi_{\text{rest}}(\Theta) = \frac{kQ}{2} \int_{\Theta}^{\infty} \left(\prod_{k=1}^D \frac{1}{\sqrt{a_k^2 + \theta}} \right) d\theta \quad (43)$$

⁵In the rest frame of the static, charged ellipsoid, we have chosen the vector potential to be identically zero. Of course, in that rest frame, any time-*independent* vector potential of the form $\vec{A}_{\text{rest}}(\vec{r}, t) = \vec{\nabla} \chi(\vec{r})$ would not alter \vec{E}_{rest} and would also give $\vec{B}_{\text{rest}} = 0$, but we choose to avoid this complication.

But again, for emphasis, for any given point \vec{r} *outside* the charged surface defined by (37), for any given t , it is now equation (38) that implicitly defines the value of Θ for the moving ellipsoidal surface that instantaneously includes the given point.

For all points \vec{r} *inside* the moving charged ellipsoid at any given time, \vec{r} must satisfy

$$\sum_{k=1}^{D-1} \frac{x_k^2}{a_k^2} + \frac{\gamma^2 (x_D - vt)^2}{a_D^2} < 1 \quad (44)$$

For such points the scalar and vector potentials are both constant, with values given by $\gamma \Phi_{\text{rest}} (\Theta = 0)$ and $\vec{v} \gamma \Phi_{\text{rest}} (\Theta = 0) / c^2$. Thus the electric and magnetic fields vanish inside the uniformly moving charged ellipsoid, as should have been expected.

Up next ...

Electromagnetic fields outside the moving ellipsoidal conductor, either from partials of Φ and \vec{A} or else by direct Lorentz transformation of the static fields.

Simple rules for the direction of the Lorentz boosted electric and magnetic fields for ellipsoids of revolution (“spheroids”).

Covariant field strength direction dependence.

Conclusions.

4 Example: charged ellipse moving in 2D

The previous results are easily illustrated and visualized for $D = 2$. At $t = 0$ the direction of the 2D electric field for a charged conducting ellipse moving along the x axis is

$$\widehat{E}(x, y) = \frac{E_1^{(0)}(\gamma x, y) \widehat{x} + \gamma E_2^{(0)}(\gamma x, y) \widehat{y}}{\sqrt{\left(E_1^{(0)}(\gamma x, y)\right)^2 + \gamma^2 \left(E_2^{(0)}(\gamma x, y)\right)^2}} \quad (45)$$

where $\overrightarrow{E^{(0)}}$ is the electric field in the rest frame of the ellipse. That is to say,

$$\widehat{E}(x, y) \propto E_1^{(0)}(\gamma x, y) \widehat{x} + \gamma E_2^{(0)}(\gamma x, y) \widehat{y} \quad (46)$$

For other t , the field has the same geometry about the point $x - vt$, i.e. the direction field (45) is simply shifted like a rigid object to the right along the x axis by an amount vt .

But, in the rest frame of a charged conducting ellipsoid of revolution (i.e. a spheroid) there is a *simple rule* for the direction of $\overrightarrow{E^{(0)}}$ at any point outside the spheroid (e.g. see [7]). The rule also applies to 2D ellipses, in which case the direction $\widehat{E^{(0)}}$ is determined by vectors $\vec{r}_1(x, y) = x_1(x, y) \widehat{x} + y_1(x, y) \widehat{y}$ and $\vec{r}_2(x, y) = x_2(x, y) \widehat{x} + y_2(x, y) \widehat{y}$ from each of the two foci to the observation point $\vec{r} = x \widehat{x} + y \widehat{y}$. (Actually, $x_{1,2}$ only depend on x , and $y_{1,2}$ only depend on y for the case at hand.) This rule is encoded in the simple analytic formula

$$\begin{aligned} \widehat{E^{(0)}}(x, y) &\propto \frac{\vec{r}_1(x, y)}{|\vec{r}_1(x, y)|} + \frac{\vec{r}_2(x, y)}{|\vec{r}_2(x, y)|} = \left(\frac{x_1(x, y)}{|\vec{r}_1(x, y)|} + \frac{x_2(x, y)}{|\vec{r}_2(x, y)|} \right) \widehat{x} + \left(\frac{y_1(x, y)}{|\vec{r}_1(x, y)|} + \frac{y_2(x, y)}{|\vec{r}_2(x, y)|} \right) \widehat{y} \\ &\propto E_1^{(0)}(x, y) \widehat{x} + E_2^{(0)}(x, y) \widehat{y} \end{aligned} \quad (47)$$

For each component the same proportionality factor appears. Therefore

$$\left(E_1^{(0)}(x, y), E_2^{(0)}(x, y) \right) \propto \left(\frac{x_1(x, y)}{|\vec{r}_1(x, y)|} + \frac{x_2(x, y)}{|\vec{r}_2(x, y)|}, \frac{y_1(x, y)}{|\vec{r}_1(x, y)|} + \frac{y_2(x, y)}{|\vec{r}_2(x, y)|} \right) \quad (48)$$

from which it follows that

$$\left(E_1^{(0)}(\gamma x, y), \gamma E_2^{(0)}(\gamma x, y) \right) \propto \left(\frac{x_1(\gamma x, y)}{|\vec{r}_1(\gamma x, y)|} + \frac{x_2(\gamma x, y)}{|\vec{r}_2(\gamma x, y)|}, \frac{\gamma y_1(\gamma x, y)}{|\vec{r}_1(\gamma x, y)|} + \frac{\gamma y_2(\gamma x, y)}{|\vec{r}_2(\gamma x, y)|} \right) \quad (49)$$

Hence there is a corresponding simple rule for the direction of the electric field at a point $\vec{r} = x \hat{x} + y \hat{y}$, for any t , outside a charged conducting ellipse when that ellipse is moving along its principal x axis with its center at $x - vt$. It is given by

$$\hat{E}(x, y, t) \propto \frac{x_1(\gamma(x - vt), y) \hat{x} + \gamma y_1(\gamma(x - vt), y) \hat{y}}{|\vec{r}_1(\gamma(x - vt), y)|} + \frac{x_2(\gamma(x - vt), y) \hat{x} + \gamma y_2(\gamma(x - vt), y) \hat{y}}{|\vec{r}_2(\gamma(x - vt), y)|} \quad (50)$$

A similar formula applies when the motion is along the y axis, namely,

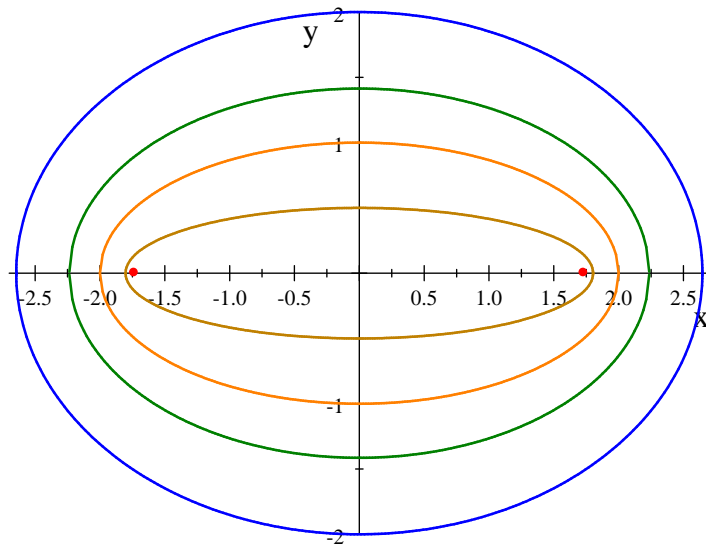
$$\hat{E}(x, y, t) \propto \frac{\gamma x_1(x, \gamma(y - vt)) \hat{x} + y_1(x, \gamma(y - vt)) \hat{y}}{|\vec{r}_1(x, \gamma(y - vt))|} + \frac{\gamma x_2(x, \gamma(y - vt)) \hat{x} + y_2(x, \gamma(y - vt)) \hat{y}}{|\vec{r}_2(x, \gamma(y - vt))|} \quad (51)$$

These same formulas work for the field of a line charge considered as a limit of an ellipse, for motion either along or perpendicular to the line of charge, respectively. (In the case where the motion is perpendicular to the line of charge, there is no Lorentz contraction of the charged line.) The accompanying graphs illustrate these geometrical features. In those graphs the RHS of (50) is called the “simplified direction field” and is plotted for $t = 0$ without being normalized to a unit vector.

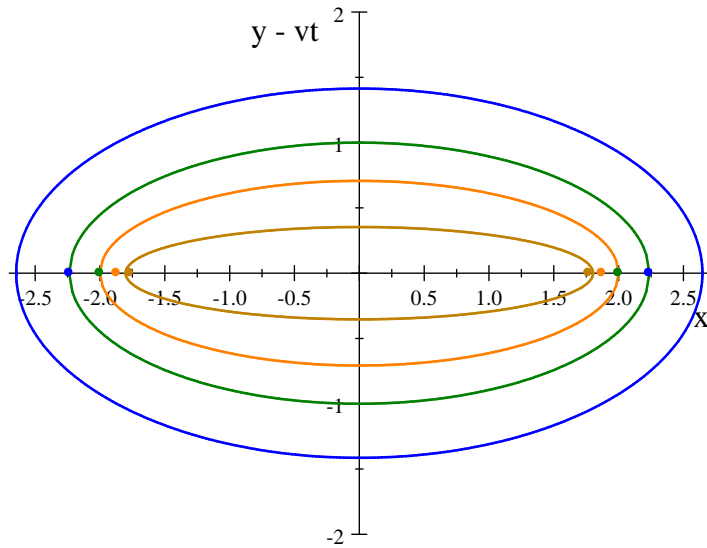
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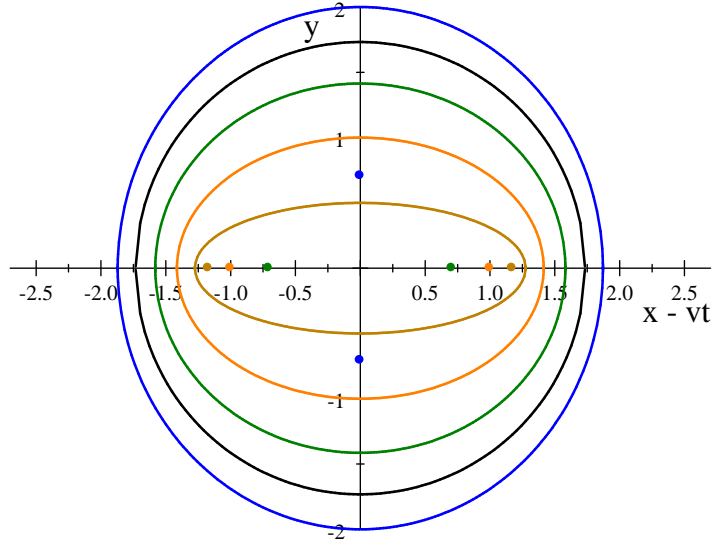
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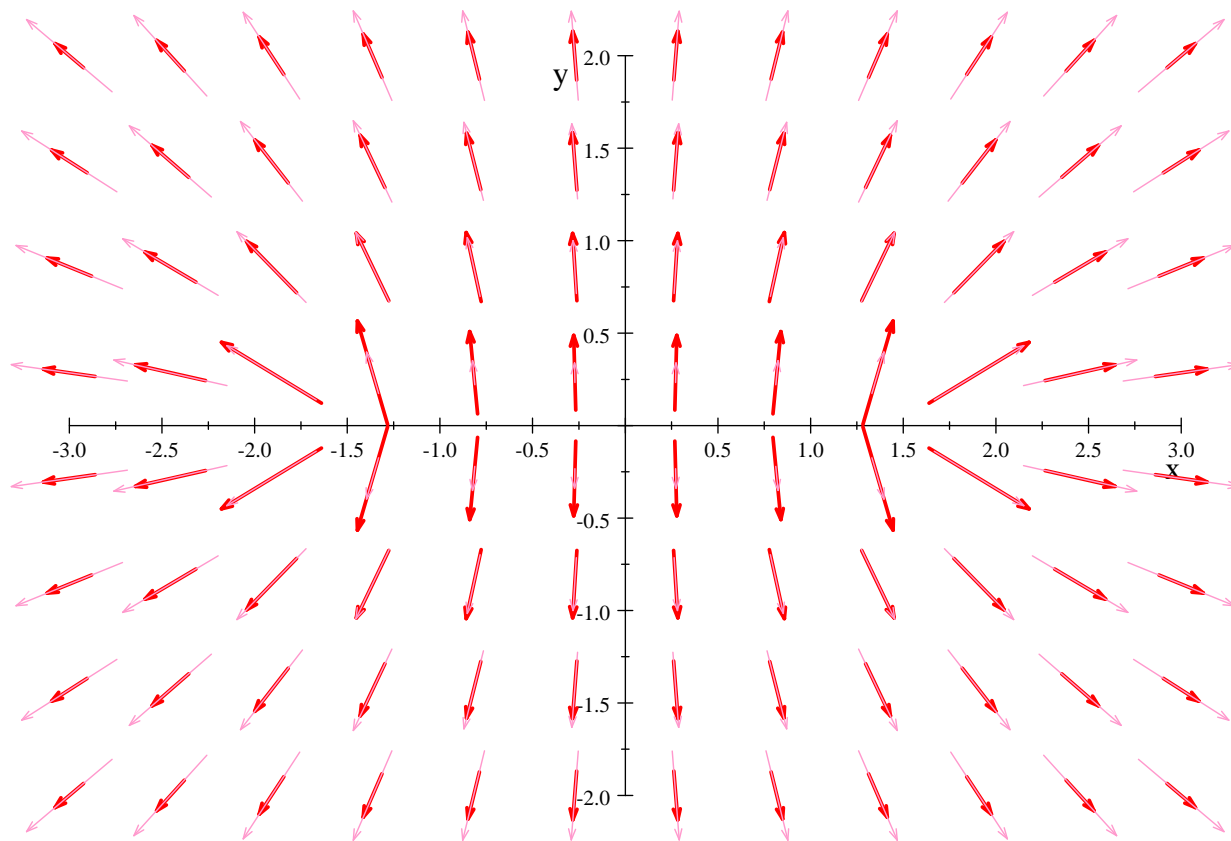
Confocal ellipses defined by $\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} = 1$ for $a = 2$, $b = 1$, and $\theta = -3/4, 0, 1, \& 3$, in sienna, orange, green, & blue, respectively. The common foci are indicated by red dots at $(x, y) = (\pm\sqrt{3}, 0)$.



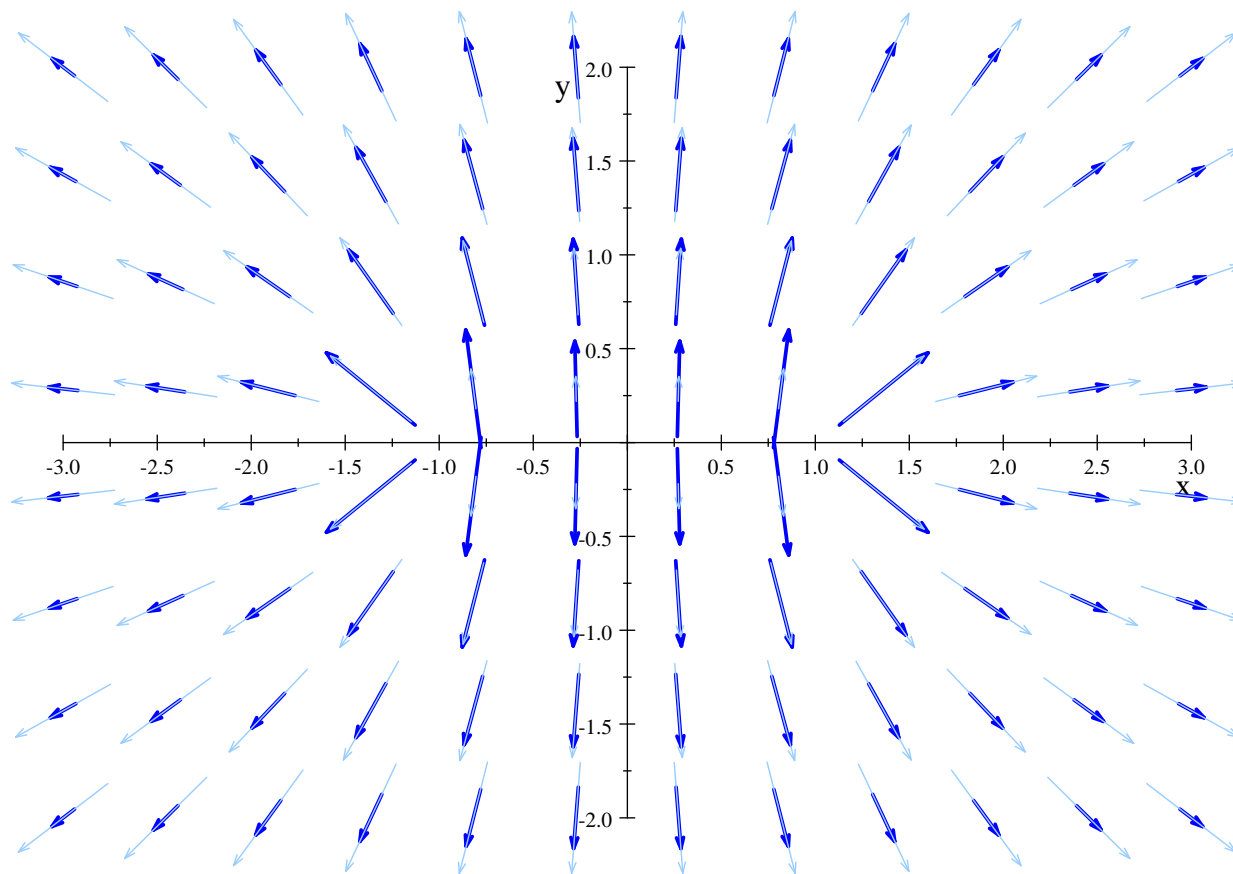
The same ellipses Lorentz contracted along the y -axis, as given by $\frac{x^2}{a^2+\theta} + \frac{\gamma^2(y-vt)^2}{b^2+\theta} = 1$, shown here for $\gamma = \sqrt{2}$. Note the vertical axis in the Figure is now $y - vt$. For each contracted ellipse the foci are indicated by dots of the corresponding color. All the foci are now farther from the origin than the $\gamma = 1$ case.



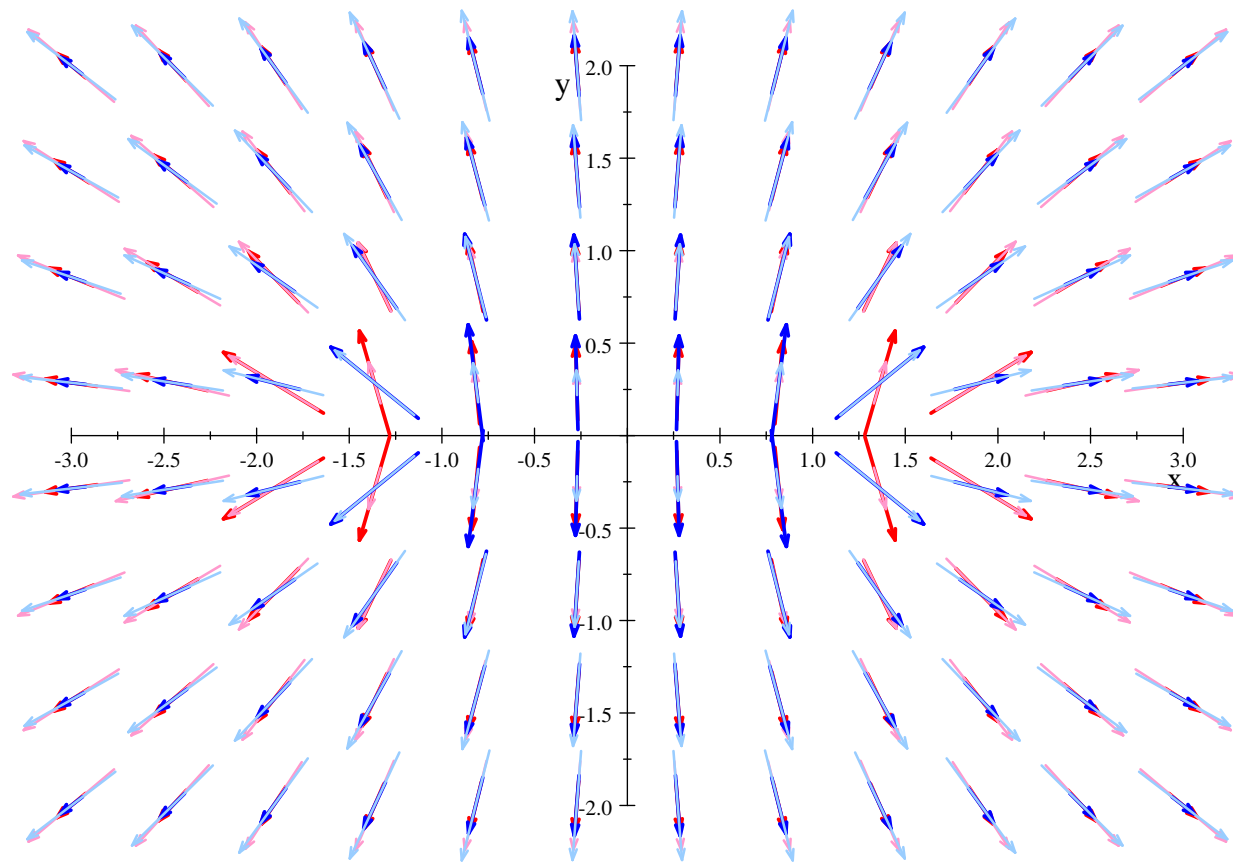
The same ellipses Lorentz contracted along the x -axis, as given by $\frac{\gamma^2(x-vt)^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} = 1$, again shown here for $\gamma = \sqrt{2}$. Note the horizontal axis in the Figure is now $x - vt$. For each contracted ellipse the foci are again indicated by dots of the corresponding color. Note for $\theta \leq 2$ the foci are now closer to the origin than the $\gamma = 1$ case. If $a > b$, when $\theta = \frac{a^2 - b^2 \gamma^2}{\gamma^2 - 1}$ for a given γ the ellipse will become a circle of radius $r = \sqrt{\frac{a^2 - b^2}{\gamma^2 - 1}}$. Here this occurs for $\theta = 2$, giving the black circle of radius $\sqrt{3}$. For larger values of θ the curves are again ellipses, but the foci are now on the y -axis, as illustrated here for $\theta = 3$.



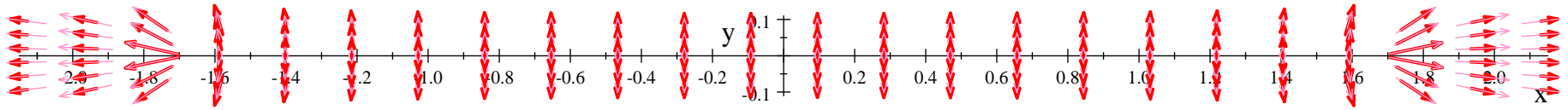
Electric field (red) and simplified direction field (pink) for a static, charged, conducting line segment in two spatial dimensions, positioned at $x \in [-\sqrt{3}, \sqrt{3}]$, $y = 0$.



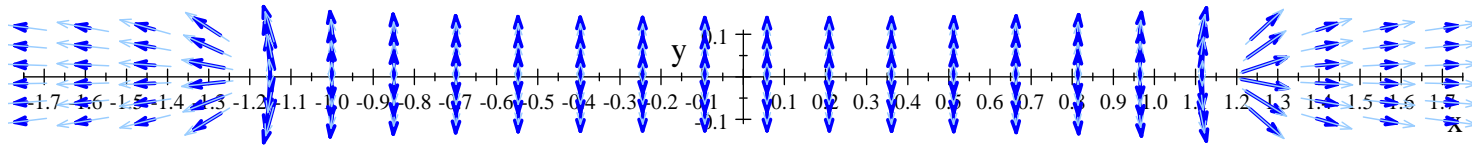
Electric field (blue) and simplified direction field (light blue) for the previous charged line segment, but now moving in the x -direction, with $\gamma = \sqrt{2}$.



Previous plots, combined.



Electric field (red) and simplified direction field (pink) for a static, charged, conducting line segment in two spatial dimensions, positioned so that $x \in [-\sqrt{3}, \sqrt{3}] = [-1.73, 1.73]$, & $y = 0$.



Electric field (blue) and simplified direction field (light blue) for the previous charged line segment, but now moving in the x -direction, with $\gamma = \sqrt{2}$, hence Lorentz contracted to have $x \in [-\sqrt{3/2}, \sqrt{3/2}] = [-1.22, 1.22]$.

Appendix: Geometry of hyperellipsoids

For an ellipsoidal “hypersurface” embedded in D dimensions, as given by $\sum_{k=1}^D x_k^2/a_k^2 = 1$, resolving the constraint by solving for $x_D(x_1, \dots, x_{D-1})$ gives simple expressions for the metric, inverse metric, and 2nd fundamental form of the hypersurface. The results are

$$g_{mn} = \delta_{mn} + \frac{a_D^4}{x_D^2} \frac{x_m x_n}{a_m^2 a_n^2} \quad \text{for } m, n = 1, 2, \dots, D-1 \quad (1)$$

$$\det g_{mn} = \frac{a_D^4}{x_D^2} S, \quad \text{where } S = \sum_{i=1}^D \frac{x_i^2}{a_i^4} > 0 \quad \text{and} \quad x_D^2 = a_D^2 \left(1 - \sum_{j=1}^{D-1} \frac{x_j^2}{a_j^2} \right) \quad (2)$$

$$g^{km} = \delta_{km} - \frac{1}{S} \frac{x_k x_m}{a_k^2 a_m^2} \quad \text{for } k, m = 1, \dots, D-1 \quad (3)$$

$$K_{mn} = -\hat{n} \cdot (\partial_m \partial_n \vec{r}) = \frac{1}{\sqrt{S}} \frac{1}{a_m^2} \left(\delta_{mn} + \frac{a_D^2}{a_n^2} \frac{x_m x_n}{x_D^2} \right) \quad \text{for } m, n = 1, \dots, D-1, \vec{r} \text{ on the hypersurface, and } \hat{n} \cdot \vec{r} = 0 \quad (4)$$

$$\mathbb{K}_{kn} \equiv g^{km} K_{mn} = \frac{1}{(S)^{3/2}} \left(\frac{S \delta_{kn}}{a_k a_n} + \frac{x_k x_n}{a_k^2 a_n^2} \left(\frac{1}{a_D^2} - \frac{1}{a_n^2} \right) \right) \quad \text{where } S = \sum_{j=1}^D \frac{x_j^2}{a_j^4} \quad \text{and} \quad x_D^2 = a_D^2 \left(1 - \sum_{i=1}^{D-1} \frac{x_i^2}{a_i^2} \right) \quad (5)$$

again for $k, m, n = 1, \dots, D-1$. Note the $\left(\frac{1}{a_D^2} - \frac{1}{a_n^2}\right)$ factor in the matrix \mathbb{K}_{kn} breaks the $k \leftrightarrow n$ symmetry. Also note the *coordinate singularity* (as opposed to a physical singularity) in the metric g_{mn} and K_{mn} on the $x_D = 0$ “equatorial” submanifold. However, there is no such singularity in \mathbb{K}_{kn} , whose eigenvalues κ_m for $m = 1, \dots, D-1$, are all *finite* so long as $a_k > 0$ for $k = 1, \dots, D$. The *intrinsic* curvature scalar densities on the manifold are encoded in

$$\det(1 + \lambda \mathbb{K}) = 1 + \lambda \left(\sum_{m=1}^{D-1} \kappa_m \right) + \lambda^2 \left(\sum_{m>n=1}^{D-1} \kappa_m \kappa_n \right) + \dots + \lambda^{D-1} \left(\prod_{m=1}^{D-1} \kappa_m \right), \quad \text{where } R = \sum_{m>n=1}^{D-1} \kappa_m \kappa_n \quad \text{etc.} \quad (6)$$

The last term in the expansion of $\det(1 + \lambda \mathbb{K})$ is $\lambda^{D-1} \det \mathbb{K}$, of course. From the above expression (5) it follows that

$$\det \mathbb{K} = \frac{1}{(\sqrt{S})^{D+1}} \left(\prod_{j=1}^D \frac{1}{a_j^2} \right) \quad (7)$$

Therefore, on a charged, conducting hyperellipsoid embedded in D dimensions, for which the hypersurface charge density is $\sigma_D = \frac{1}{\Omega_D \sqrt{S}} \prod_{j=1}^D \frac{1}{a_j}$, it follows that $\sigma^{D+1} \propto \det \mathbb{K}$, or more precisely

$$(\Omega_D \sigma_D)^{D+1} = \left(\prod_{j=1}^D \frac{1}{a_j^{D-1}} \right) \det \mathbb{K} = \left(\prod_{j=1}^D \frac{1}{a_j^{D-1}} \right) \left(\prod_{m=1}^{D-1} \kappa_m \right) \quad (8)$$

Here $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$ is the surface area of a unit radius sphere embedded in D dimensions (i.e. the total “solid” angle around the center of the sphere).