PHY752, Fall 2016, Assigned Problems

For clarification or to point out a typo (or worse!) please send email to curtright@miami.edu

[0] Find the URL for the course webpage and email it to *curtright@miami.edu* (just the URL, please).

[1] Current $I(t)$ flows along an ideal line (e.g. the negative z axis, with $I(t) > 0$ for current flowing towards $z = 0$) and terminates at a point (e.g. the origin of coordinates) where it accumulates to produce charge $Q(t)$. Assume that charge is *not* accumulated at any *other* point. (a) Find the relation in general between $I(t)$ and $Q(t)$, and in particular find $Q(t)$ when the current has linear time dependence, $I(t) = I_0 + K_0 t$ where I_0 and K_0 are constants. (b) Write expressions for $\rho(\overrightarrow{r}, t)$ and $\overrightarrow{J}(\overrightarrow{r}, t)$ making use of Dirac deltas and the Heaviside step function Θ, both for general $I(t)$ & $Q(t)$ and for the particular case when the current is linear in time. (c) For such an ideal current flow, simplify the integrals in the Schott–Panofsky-Phillips–Jefimenko expressions as much as possible, and explicitly evaluate the integrals when the current is linear in time to obtain $\overrightarrow{B}(\overrightarrow{r}, t)$ and $\overrightarrow{E}(\overrightarrow{r}, t)$ for all points that avoid the current and charge. If you encounter a divergent integral, do something to fix it!

[2] For any given *localized* charge and current densities, it is possible to define Φ causally with respect to *any speed* v!!! Namely,

$$\Phi_{v\text{-gauge}}(\overrightarrow{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{1}{|\overrightarrow{r} - \overrightarrow{s}|} \, \rho(\overrightarrow{s}, t_<) \, d^3 s \tag{1}$$

where the "v-dependent retarded time" at the source point is now $t_< \equiv t - |\overrightarrow{r} - \overrightarrow{s}| / v$. What inhomogeneous, second-order, partial differential equation does $\Phi_{v\text{-gauge}}$ obey? Construct a gauge transformation $\Omega(\overrightarrow{r}, t)$ that connects this v-gauge to the Lorenz gauge. Work out an explicit expression for the vector potential $\overrightarrow{A}_{v\text{-gauge}}(\overrightarrow{r}, t)$ in this class of gauges. What inhomogeneous, second-order, partial differential equation does $\overrightarrow{A}_{v\text{-gauge}}$ obey? Show that the v-gauge potentials satisfy the constraint

$$\overrightarrow{\nabla} \cdot \overrightarrow{A}_{v\text{-gauge}} + \frac{1}{v^2} \partial_t \Phi_{v\text{-gauge}} = 0 \tag{2}$$

and take note of the Coulomb ($v \to \infty$) and Lorenz gauge ($v \to c$) limits of all your results.

[3] Suppose an electric field $\overrightarrow{E}(\overrightarrow{r})$ is due entirely to charge distributed on a *planar* surface, with surface normal $\widehat{n}$, and with local charge/area on the surface given by $\sigma(\overrightarrow{s})$, where $\overrightarrow{s}$ is a point on the surface. Without loss of generality, consider an observation point $\overrightarrow{r}$ "above the surface" (e.g. choose the origin of coordinates on the surface, so that $\widehat{n} \cdot \overrightarrow{r} > 0$ for points above the surface). Express $\widehat{n} \cdot d\overrightarrow{E}(\overrightarrow{r})$ in terms of $\sigma(\overrightarrow{s}) \; d\Omega(\overrightarrow{r}, \overrightarrow{s})$ where $d\Omega(\overrightarrow{r}, \overrightarrow{s})$ is the solid angle subtended by the infinitesimal surface area element located at point $\overrightarrow{s}$ on the surface, $\overrightarrow{dArea}(\overrightarrow{s})$, as measured from the perspective of the observation point $\overrightarrow{r}$, and where $d\overrightarrow{E}(\overrightarrow{r})$ is the infinitesimal electric field due to the charge only on $\overrightarrow{dArea}(\overrightarrow{s})$. Write $\widehat{n} \cdot \overrightarrow{E}(\overrightarrow{r})$ in terms of an integral over the charged surface, such as $\int \sigma(\overrightarrow{s}) \, d\Omega(\overrightarrow{r}, \overrightarrow{s})$. What is the limit as the observation point approaches a point on the surface, $\lim_{\overrightarrow{r} \to \overrightarrow{s}} \widehat{n} \cdot \overrightarrow{E}(\overrightarrow{r})$? If the charge distribution is bounded for all points on the surface, $\sigma_{\min} \leq \sigma(\overrightarrow{s}) \leq \sigma_{\max}$, show that

$$\frac{\sigma_{\min}}{4\pi\varepsilon_0} \, \Omega_{\text{total}} \leq \widehat{n} \cdot \overrightarrow{E}(\overrightarrow{r}) \leq \frac{\sigma_{\max}}{4\pi\varepsilon_0} \, \Omega_{\text{total}} \; , \tag{3}$$

where Ω_{total} is the total solid angle subtended by the surface, as measured from $\overrightarrow{r}$. What does this imply if $\sigma(\overrightarrow{s})$ is constant on the surface?

[4] Find exact algebraic expressions for the curves in the xy-plane that describe the electric field lines for a 3D point dipole $\overrightarrow{p} = p \, \widehat{x}$ located at the origin. Compare your answer to curves given by

$$x^2 + y^2 = C \; y^{4/3} \tag{4}$$

for various positive constants C. Plot some of the electric field lines in the upper half plane $y > 0$.

[5] Compare potentials at various points in three dimensions to appreciate how uniform charge distributions are *not* equipotentials, in general. For example: For a uniform sphere of charge, show that

$$\Phi_{\text{center}} = \frac{3}{2}\ \Phi_{\text{surface}}\ . \tag{5}$$

Plot the potential for all r, where $r = \sqrt{x^2 + y^2 + z^2}$ is the distance from the center of the sphere. For a flat, uniformly charged, circular disk of radius R, show that

$$\Phi_{\text{center}} = \frac{\pi}{2}\ \Phi_{\text{edge}}\ . \tag{6}$$

Plot the potential on the surface of the disk as a function of the distance from the center of the disk, $r = \sqrt{x^2 + y^2}$, for all $r \leq R$. For a uniformly charged rectangular parallelepiped, with side lengths a, b, and c, show that

$$\Phi_{\text{center}} = 2\ \Phi_{\text{corner}}\ . \tag{7}$$

(Hint: In this last case, use dimensional analysis and linear superposition to avoid having to do any integrals!) Does your result change if the parallelepiped is not rectangular? For a uniformly charged right circular cylinder, of length L and radius R, how does

$$\Phi_{\text{center of cylinder}}/\Phi_{\text{center of circular end face}} \tag{8}$$

depend on L and R? What can you say about $\Phi_{\text{center of cylinder}}/\Phi_{\text{edge of circular end face}}$?

[6] There are five "Platonic solids" in 3D, as described here, https://en.wikipedia.org/wiki/Platonic_solid. For the purposes of this problem you should consider these five shapes to be hollow surfaces. For each of these five cases, if only one of the faces is at a constant potential Φ_0 while all the other faces are grounded (i.e. at potential $\Phi = 0$), what is the potential at the exact center of the hollow surface?

[7] This problem and the next are some elementary exercises from vector calculus in *three* dimensions. First, some derivatives. Consider a scalar function $\Phi(\overrightarrow{r})$. It turns out that angular derivatives of $\Phi(\overrightarrow{r})$ may be expressed using the differential operators $\overrightarrow{L} = -i\overrightarrow{r} \times \overrightarrow{\nabla}$ and $L^2 = \overrightarrow{L} \cdot \overrightarrow{L}$. Show the following.

$$\begin{aligned}
\nabla^2\Phi &= \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\Phi) - \frac{1}{r^2}\ L^2\Phi\ , &(9)\\
\overrightarrow{\nabla} \times \overrightarrow{L}\ \Phi &= -i\overrightarrow{r}\ \nabla^2\Phi + i\left(2 + \overrightarrow{r}\cdot\overrightarrow{\nabla}\right)\overrightarrow{\nabla}\Phi\ , &(10)\\
r^2\overrightarrow{\nabla} \times \overrightarrow{L}\ \Phi &= \overrightarrow{r} \times \overrightarrow{L}\ \left(1 + \overrightarrow{r}\cdot\overrightarrow{\nabla}\right)\Phi + i\overrightarrow{r}\ L^2\Phi\ . &(11)
\end{aligned}$$

Also, compute $\overrightarrow{\nabla} \cdot \left(f(\theta,\phi)\ \widehat{r}/r^2\right)$ for any angular function $f(\theta,\phi)$.

[8] Some more vector calculus exercises in three dimensions. Integrals this time. For a 3D vector of *fixed* radius $r = |\overrightarrow{r}|$, show that

$$\begin{aligned}
\int r_j r_k\ d\Omega &= \frac{4\pi}{3}\ \delta_{jk}r^2\ , &(12)\\
\int r_j r_k r_l r_m\ d\Omega &= \frac{4\pi}{15}\ \left(\delta_{jk}\delta_{lm} + \delta_{jl}\delta_{km} + \delta_{jm}\delta_{kl}\right)r^4\ , &(13)
\end{aligned}$$

etc., where r_j for $j = 1, 2, 3$ are the rectangular Cartesian components of the vector $\overrightarrow{r}$ (i.e. $r_1 = x$, $r_2 = y$, and $r_3 = z$) and where the integration $\int d\Omega$ is over all angular *directions* for the vector $\overrightarrow{r}$. What do you obtain for $\int r_j\ d\Omega$ and $\int r_j r_k r_l\ d\Omega$?

Given Gauss' theorem for a spatial volume $\mathcal{V}$ with boundary surface $\mathcal{A}$ (i.e. $\int_{\mathcal{V}} \overrightarrow{\nabla}\cdot\overrightarrow{V} d^3r = \int_{\mathcal{A}} \overrightarrow{V}\cdot\widehat{n}\ d^2s$ where $\widehat{n}$ is the local normal to the surface $\mathcal{A}$), show that

$$\int_{\mathcal{V}} r_j r_k \overrightarrow{\nabla}\cdot\overrightarrow{V} d^3r = \int_{\mathcal{A}} r_j r_k \overrightarrow{V}\cdot\widehat{n}\ d^2s - \int_{\mathcal{V}} \left(r_j V_k + r_k V_j\right) d^3r \tag{14}$$

where $\overrightarrow{V}$ is any vector that is sufficiently well-behaved so that the integrals are defined. Extend this result to integrals of the form $\int_{\mathcal{V}} r_j r_k \cdots r_m \overrightarrow{\nabla}\cdot\overrightarrow{V} d^3r$.

[9] The Yukawa potential, Ψ, is given by a solution to the inhomogeneous Helmholtz equation,

$$\left(\nabla^2 - \frac{1}{\ell^2}\right)\Psi\left(\overrightarrow{r}\right) = -\frac{1}{\varepsilon_0}\,\rho\left(\overrightarrow{r}\right)\ , \tag{15}$$

where ℓ is a fixed length, as discussed in your text and in class lecture. Consider two ideal concentric conducting spherical shells, of radii $r_1 < r_2$, each of negligible thickness. The two shells are arranged to be at a common potential Ψ_0. When $\frac{1}{\ell} = 0$, as is well known, all charge will reside on the outer shell. However, for finite $\ell > 0$ some charge will be found on the inner shell. Show to leading order for large ℓ that the charge on the inner shell is given by

$$Q = \frac{2\pi\varepsilon_0\Psi_0}{3}\left(\frac{r_1 r_2}{\ell^2}\right)(r_1 + r_2) + O\left(\frac{1}{\ell^3}\right) \quad \text{for} \quad \ell \gg r_2 > r_1\ . \tag{16}$$

[10] Suppose the potential of a point charge q is given by a modified power law:

$$\chi(r) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r^{1+\delta}} \quad \text{for} \quad 0 < \delta < 1\ . \tag{17}$$

For a uniformly charged spherical shell of radius R, and surface charge density $\sigma = \frac{Q}{4\pi R^2}$, use linear superposition and explicit integration to find χ for both $r < R$ and $r > R$.

[11] Use the eigenfunction method to determine the Dirichlet boundary condition Green function satisfying $\nabla^2 G\left(\overrightarrow{r}, \overrightarrow{s}\right) = -4\pi\delta^3\left(\overrightarrow{r} - \overrightarrow{s}\right)$ for the three dimensional half-space $-\infty < y, z < +\infty$, $0 \leq x < \infty$, with the homogeneous boundary conditions $G(x = 0) = 0$ with G bounded at spatial infinity. Begin by finding a complete set of eigenfunctions $\Psi_{\overrightarrow{k}}\left(\overrightarrow{r}\right)$ that satisfy the boundary condition. What are the allowed eigenvalues $\lambda\left(\overrightarrow{k}\right)$ such that $\nabla^2\Psi_{\overrightarrow{k}}\left(\overrightarrow{r}\right) = -\lambda\left(\overrightarrow{k}\right)\ \Psi_{\overrightarrow{k}}\left(\overrightarrow{r}\right)$? Perform all the necessary sums or integrals to obtain your answer for $G\left(\overrightarrow{r}, \overrightarrow{s}\right)$ in a simple closed form. Compare your result to the Green function obtained by the method of images. [Hint: For the full space $-\infty < x, y, z < +\infty$, the familiar Green function that vanishes at spatial infinity is given by

$$\frac{1}{\left|\overrightarrow{r} - \overrightarrow{s}\right|} = \frac{1}{2\pi^2}\int_{-\infty}^{+\infty} dk_x \int_{-\infty}^{+\infty} dk_y \int_{-\infty}^{+\infty} dk_z\ \frac{e^{i\overrightarrow{k}\cdot\left(\overrightarrow{r} - \overrightarrow{s}\right)}}{k_x^2 + k_y^2 + k_z^2} \tag{18}$$

where the completeness of the eigenfunctions is expressed as $\frac{1}{(2\pi)^3}\iiint_{-\infty}^{+\infty} e^{i\overrightarrow{k}\cdot\left(\overrightarrow{r} - \overrightarrow{s}\right)} d^3k = \delta^3\left(\overrightarrow{r} - \overrightarrow{s}\right)$. Now, mind the boundary condition at $x = 0$ in the problem for the half-space.]

[12] Use either the method of images or the eigenfunction method (or both!) to find a *Neumann* boundary condition Green function for the three dimensional half-space $-\infty < y, z < +\infty$, $0 \leq x < \infty$, with the homogeneous boundary condition $\partial_x G(x = 0) = 0$. Use this Green function to give an integral expression for Φ when there is only a surface charge density $\sigma\left(\overrightarrow{s}\right)$ on the $x = 0$ plane (i.e. away from the plane, for $x > 0$, there is no additional charge). Show that your integral expression for the potential in this situation reduces to the expected Coulomb expression, namely,

$$\Phi\left(\overrightarrow{r}\right) = \frac{1}{4\pi\varepsilon_0}\int \frac{\sigma\left(\overrightarrow{s}\right)}{\left|\overrightarrow{r} - \overrightarrow{s}\right|}\ d^2s \tag{19}$$

where the integral is over all points $\overrightarrow{s}$ on the $x = 0$ plane.

[13] The Green function for a cube with $0 \leq x, y, z \leq a$, with the boundary condition that $G = 0$ on all six sides of the cube, can be written as a triple sum:

$$G\left(\overrightarrow{r}, \overrightarrow{s}\right) = \frac{32}{\pi a}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\sin(l\pi x/a)\sin(l\pi u/a)\sin(m\pi y/a)\sin(m\pi v/a)\sin(n\pi z/a)\sin(n\pi w/a)}{l^2 + m^2 + n^2}\ , \tag{20}$$

where $\overrightarrow{r} = (x, y, z)$ and $\overrightarrow{s} = (u, v, w)$ are the observation and source points, each located inside the cube. This Green function is normalized so that

$$\nabla^2 G\left(\overrightarrow{r}, \overrightarrow{s}\right) = -4\pi\delta^3\left(\overrightarrow{r} - \overrightarrow{s}\right)\ . \tag{21}$$

One of the sums, say the sum on n, can be performed to obtain an elementary function. Thus G may also be written as a double sum:

$$G(\vec{r},\vec{s}) = \frac{16}{a}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{\sin\left(\frac{l\pi x}{a}\right)\sin\left(\frac{l\pi u}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sin\left(\frac{m\pi v}{a}\right)\sinh\left(\frac{\pi z_<}{a}\sqrt{l^2+m^2}\right)\sinh\left(\frac{\pi(a-z_>)}{a}\sqrt{l^2+m^2}\right)}{\sqrt{l^2+m^2}\sinh\left(\pi\sqrt{l^2+m^2}\right)} \tag{22}$$

where $z_{\gtrless} = \max/\min\ (z,w)$. Use this G to determine the potential $\Phi(x,y,z)$ everywhere within an *empty* cube whose upper surface (i.e. $z=a$) is at a constant potential $\Phi = V$, and whose other five sides are "grounded" with $\Phi = 0$. Your answer will involve either the triple sum, if (20) is used for G, or the double sum, if (22) is used for G. Given the boundary conditions, do the sums simplify? By truncating the sums at reasonable upper limits, use your results to find a numerical estimate for the value of Φ/V at the center of the cube, $(x,y,z) = (a/2,a/2,a/2)$. (You might take note that the double sum result usually evaluates much quicker than the triple sum result, for numerical purposes.) How does your estimate for $\Phi(a/2,a/2,a/2)/V$ compare to $1/6$?

[14] The homogeneous Dirichlet boundary condition Green function for the region between two concentric circular cylinders centered on the z-axis, with radii $a<b$, and with planar end caps at $z=0$ and L, is given by

$$\begin{aligned} G(\vec{r},\vec{s}) &= \frac{4}{L}\sum_{m=-\infty}^{+\infty} e^{im(\phi-\varphi)}\sum_{n=1}^{\infty}\frac{\sin(n\pi z/L)\sin(n\pi w/L)}{K_m(n\pi a/L)\,I_m(n\pi b/L) - K_m(n\pi b/L)\,I_m(n\pi a/L)} \\ &\quad\times\left(I_m(n\pi r_</L)\,K_m(n\pi a/L) - I_m(n\pi a/L)\,K_m(n\pi r_</L)\right) \\ &\quad\times\left(I_m(n\pi b/L)\,K_m(n\pi r_>/L) - I_m(n\pi r_>/L)\,K_m(n\pi b/L)\right) \end{aligned} \tag{23}$$

where in cylindrical coordinates, $\vec{r} = (r,\phi,z)$, $\vec{s} = (s,\varphi,w)$, and $r_{\lessgtr} = \min/\max(r,s)$. Make some checks to see if this series solution for the Green function satisfies

$$\nabla^2 G(\vec{r},\vec{s}) = -\,4\pi\,\delta^3(\vec{r}-\vec{s}) = -\,\frac{4\pi}{r}\,\delta(r-s)\,\delta(\phi-\varphi)\,\delta(z-w) \tag{24}$$

as well as $G=0$ on the cylinder boundaries.

[15] The free space Green function may be written as a linear combination of Bessel function bilinears. For example, as discussed in class,

$$\frac{1}{|\vec{r}-\vec{s}|} = \sum_{m=-\infty}^{+\infty} e^{im(\phi-\varphi)}\int_0^{\infty} J_m(kr)\,J_m(ks)\,e^{-k|z-w|}dk \tag{25}$$

where in cylindrical coordinates, $\vec{r} = (r,\phi,z)$ and $\vec{s} = (s,\varphi,w)$. Moreover, in this case the sum on m may be evaluated in closed form to obtain

$$\sum_{m=-\infty}^{+\infty} e^{im(\phi-\varphi)}J_m(kr)\,J_m(ks) = J_0\left(k\sqrt{r^2+s^2-2rs\cos(\phi-\varphi)}\right) \tag{26}$$

(This is a special case of Graf's addition theorem for Bessel functions.) Therefore it is also true that

$$\frac{1}{|\vec{r}-\vec{s}|} = \int_0^{\infty} J_0\left(k\sqrt{r^2+s^2-2rs\cos(\phi-\varphi)}\right)e^{-k|z-w|}dk \tag{27}$$

Verify this last result by using the integral representation

$$J_0(w) = \frac{1}{\pi}\int_0^{\pi} e^{iw\cos\theta}d\theta \tag{28}$$

as well as the elementary integral: $\int_0^{\pi}\frac{1}{a-ib\cos\theta}d\theta = \frac{\pi}{\sqrt{a^2+b^2}}$ for real $a>0$ and real b.

[16] The potential surrounding an ideal equipotential line segment of length L, lying along the x-axis and centered on the origin in a two-dimensional space, can be expanded as

$$\Phi(x,y) = \int_0^{\infty} A(k)\cos kx\ e^{-k|y|}dk \tag{29}$$

where the coefficients $A(k)$ are to be determined. The mixed boundary conditions $\Phi(-L/2 \le x \le L/2, y = 0) = V$, a constant, and $\partial_y \Phi(x > L/2 \text{ or } x < -L/2, y = 0) = 0$ give a "dual" pair of integral equations to be solved for the coefficients $A(k)$, namely,

$$\int_0^\infty A(k) \cos kx \; dk = V \quad \text{for} \quad -L/2 \le x \le L/2 \tag{30}$$

$$\int_0^\infty k A(k) \cos kx \; dk = 0 \quad \text{for} \quad |x| > L/2 \tag{31}$$

Find $A(k)$ that solves this pair of equations.

[17] For a uniformly polarized finite volume of dielectric material, surrounded by empty space, the resulting static electric field is given by Coulomb's law for a surface charge density $\sigma_P(\vec{s}) = \vec{P} \cdot \hat{n}(\vec{s})$, where $\hat{n}(\vec{s})$ is the normal unit vector at the point $\vec{s}$ on the surface S that bounds the volume V, and where $\vec{P}$ is the constant polarization density within V. That is to say,

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\vec{r} - \vec{s}}{|\vec{r} - \vec{s}|^3} \; \vec{P} \cdot \hat{n}(\vec{s}) \; d^2 s \tag{32}$$

Show that this can be re-written as

$$\vec{E}(\vec{r}) = -\vec{P} \cdot \vec{\nabla} \left(\frac{1}{4\pi\varepsilon_0} \int_S \frac{\vec{r} - \vec{s}}{|\vec{r} - \vec{s}|^3} \; d^2 s \right)$$

This result is often called Poisson's formula.

[18] A spherical shell of isotropic, homogeneous, linear dielectric material, with permeability $\varepsilon > \varepsilon_0$, surrounds a spherical conductor. The dielectric shell and the conductor are concentric with a common center. The conductor has radius a and carries a "free" charge Q. The shell of dielectric has radii b and c, with $c > b > a$, and has no net charge. Determine $\Phi(\vec{r})$, $\vec{E}(\vec{r})$, and $\vec{D}(\vec{r}) = \varepsilon \vec{E}(\vec{r})$ for the four regions: (i) $r > c$, (ii) $c > r > b$, (iii) $b > r > a$, and (iv) $a > r$. Also determine the total induced charges on the dielectric surfaces at $r = b$ and $r = c$. What is the induced "space charge" $\rho_P = -\vec{\nabla} \cdot \vec{P}$ inside the dielectric? What do your results become in the "conductor limit" $\varepsilon_0/\varepsilon \to 0$?

[19] Use symmetry arguments and the integral form of Ampere's law to determine the magnetic field outside and inside a right circular cylinder, of radius a, carrying a uniform (constant) current density $\vec{J}$ directed along the axis of the cylinder. Find the Coulomb gauge vector potential for this same current configuration by solving the differential equation

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \; . \tag{33}$$

Again, use symmetry arguments to simplify the problem. Show that your two results agree, namely, that $\vec{B} = \vec{\nabla} \times \vec{A}$.

[20] Determine $\vec{B}$ on the axis of a trapezoidal cylinder of revolution which carries constant current/length $\vec{K} = K\,\hat{\varphi}$ around the circular sides of the cylinder. Consider the limit of a right circular cylinder as a check on your result. Also consider the "flat washer" limit of the trapezoid.

[21] An ideal straight line "wire" segment aligned with the z axis carries current I from $z = a$ to $z = b > a$. At the upper end the wire terminates on an ideal conducting plane which is parallel to the x and y axes. Similarly, the wire terminates at its lower end on another ideal conducting plane parallel to the first. On the upper plane there is a surface current density given by $\vec{K} = I\,\hat{\rho}/(2\pi\rho)$ where $\vec{\rho} = x\,\hat{x} + y\,\hat{y}$. On the lower plane there is a current density given by $-\vec{K}$. The magnetic field vanishes as $r \to \infty$. Determine $\vec{B}$ everywhere.

[22] An infinite slab of ferromagnetic material fills the space between two planes at $z = 0$ and $z = h$, with both planes parallel to the x and y axes. The material is uniformly magnetized with constant density $\vec{M}$

that makes an angle θ with respect to the z axis. There is empty space above and below the slab. Compute the effective magnetic surface charge and current densities, σ_M and $\overrightarrow{K}_M$, on both the top and bottom of the slab, and determine $\overrightarrow{B}$ and $\overrightarrow{H}$ everywhere.

[23] An ellipsoid of revolution is defined by $\left(x^2+y^2\right)/a^2+z^2/c^2=1$. The surface of the ellipsoid is charged such that it is an electrostatic equipotential. The surface charge density on the ellipsoid is

$$\sigma\left(x,y,z\right)=\frac{Q}{4\pi a^2 c}\frac{1}{\sqrt{\left(x^2+y^2\right)/a^4+z^2/c^4}}\ , \tag{34}$$

where Q is the total charge on the surface. Suppose the ellipsoid is now rigidly rotating with angular velocity ω about the z axis, with the same surface charge density. (We sprayed the surface with Krazy® Glue to keep the charges in place before it began to rotate!) Determine the resulting magnetic field $\overrightarrow{B}\left(z\right)$ everywhere along the z axis.

On the other hand, if the ellipsoid were stationary and uncharged, but filled instead with a magnetized material, determine a magnetization density $\overrightarrow{M}\left(\overrightarrow{r}\right)$ that would produce the same magnetic field, $\overrightarrow{B}\left(z\right)$. Also determine $\overrightarrow{H}\left(z\right)$ in this situation.