PHY753, Spring 2020, Assigned Problems

For clarification or to point out a typo (or worse!) please send email to curtright@miami.edu

[0] Find the URL for the course webpage and email it to *curtright@miami.edu* (just the URL, please).

[1] An infinitely long, solid cylindrical conductor carries a constant current in the direction of its axis, distributed uniformly over its circular cross sectional area, as described by Ohm's law: $\overrightarrow{J} = \sigma \overrightarrow{E}$. More specifically, $\overrightarrow{J} = \sigma E\, \widehat{z}$, where σ is a constant, where E is constant and nonzero within the cylinder but zero outside, and where we have taken the symmetry axis of the cylinder to be the z axis. Compute $\overrightarrow{B}(r)$, where r here is the length of the $\perp$ from the axis of the cylinder to the point in question. Compute $\overrightarrow{S} = \overrightarrow{E} \times \overrightarrow{H}$ at all points within or on the surface of the conducting cylinder. Compute the elementary "I^2R" power loss within any concentric *sub*-cylinder of length L and radius r, and compare it to the electromagnetic energy flux through the surface of the same sub-cylinder.

[2] Consider two concentric equipotential spherical surfaces of radii a and b, with potential difference V, and with the region between the two spheres filled with a conducting material whose conductivity is σ. If total current I were to flow uniformly and symmetrically from the inner sphere to the outer sphere, such that $\overrightarrow{J} = I\widehat{r}/\left(4\pi r^2\right)$, then according to Ohm's law, what is $\overrightarrow{E}(\overrightarrow{r})$? What is the effective resistance R of this spherical configuration, such that $V = IR$? What is the power "lost" as mechanical heat in the resistive material in this case? On the other hand, what is $\overrightarrow{B}(\overrightarrow{r})$ for such a spherically symmetric current flow, and what is the Poynting vector in this situation? Considering your answer for $\overrightarrow{S}$, in terms of electromagnetic energy how do you account for the power lost as heat in this spherical configuration?

[3] Current $I(t)$ flows along an ideal line (e.g. the negative z axis, with $I(t) > 0$ for current flowing towards $z = 0$) and terminates at a point (e.g. the origin of coordinates) where it accumulates to produce charge $Q(t)$. Assume that charge is *not* accumulated at any *other* point. (a) Find the relation in general between $I(t)$ and $Q(t)$, and in particular find $Q(t)$ when the current has linear time dependence, $I(t) = I_0 + K_0 t$ where I_0 and K_0 are constants. (b) Write expressions for $\rho(\overrightarrow{r}, t)$ and $\overrightarrow{J}(\overrightarrow{r}, t)$ making use of Dirac deltas and the Heaviside step function Θ, both for general $I(t)$ & $Q(t)$ and for the particular case when the current is linear in time. (c) For such an ideal current flow, simplify the integrals in the Schott–Panofsky-Phillips–Jefimenko expressions as much as possible, and explicitly evaluate the integrals when the current is linear in time to obtain $\overrightarrow{B}(\overrightarrow{r}, t)$ and $\overrightarrow{E}(\overrightarrow{r}, t)$ for all points that avoid the current and charge. If you encounter a divergent integral, do something to fix it!

[4] Show for a uniformly moving point charge that $\overrightarrow{E}$ and $\overrightarrow{B}$ obey *all* of Maxwell's equations, where $\rho(\overrightarrow{r}, t) = q\, \delta^3(\overrightarrow{r} - \overrightarrow{v}t)$ and $\overrightarrow{J}(\overrightarrow{r}, t) = \overrightarrow{v}\, \rho(\overrightarrow{r}, t)$. Note that the coordinates in these charge and current densities are those of the observer for whom the charge's trajectory is $\overrightarrow{\mathfrak{r}}(t) = \overrightarrow{v}t$.

[5] For any given *localized* charge and current densities, it is possible to define Φ causally with respect to *any* v of your choosing!!! Namely,

$$\Phi_{v\text{-gauge}}(\overrightarrow{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{1}{|\overrightarrow{r} - \overrightarrow{s}|}\, \rho\left(\overrightarrow{s}, t - \frac{1}{v}|\overrightarrow{r} - \overrightarrow{s}|\right) d^3s \tag{1}$$

where the "v-dependent retarded time" at the source point is now $t - |\overrightarrow{r} - \overrightarrow{s}|/v$. What inhomogeneous, second-order, partial differential equation does $\Phi_{v\text{-gauge}}$ obey? Construct a gauge transformation $\chi(\overrightarrow{r}, t)$ that connects this v-gauge to the Lorenz gauge. Work out an explicit expression for the vector potential $\overrightarrow{A}_{v\text{-gauge}}(\overrightarrow{r}, t)$ in this class of gauges. What inhomogeneous, second-order, partial differential equation does $\overrightarrow{A}_{v\text{-gauge}}$ obey? Show that the v-gauge potentials satisfy the constraint

$$\overrightarrow{\nabla} \cdot \overrightarrow{A}_{v\text{-gauge}} + \frac{1}{v^2}\, \partial_t \Phi_{v\text{-gauge}} = 0 \tag{2}$$

and take note of the Coulomb ($v \to \infty$) and Lorenz gauge ($v \to c$) limits of all your results.

[6] From the Lienard-Wiechert potentials Φ and $\vec{A}$ for a moving point particle, show that the electric field is given by:

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{R^2}\,\frac{1}{\left(1-\hat{R}\cdot\vec{v}/c\right)^3}\left[\left(1-v^2/c^2\right)\left(\hat{R}-\vec{v}/c\right)+\frac{1}{c^2}\vec{R}\times\left(\left(\hat{R}-\vec{v}/c\right)\times\vec{a}\right)\right]\Bigg|_{t_<} \tag{3}$$

where the particle's trajectory determines $\vec{\mathfrak{r}}(t)$, $\vec{v}(t) = d\vec{\mathfrak{r}}(t)/dt$, and $\vec{a}(t) = d^2\vec{\mathfrak{r}}(t)/dt^2$, where $\vec{R}(t) = \vec{r} - \vec{\mathfrak{r}}(t)$, and where *all* quantities on the RHS of (3) are evaluated at the earlier time $t_<$ defined implicitly by $t_< = t - R(t_<)/c$. Show equivalence between (3) and the Feynman expression for the electric field of a moving point charge, namely,

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\left[\left(1+\frac{R(t_<)}{c}\frac{d}{dt}\right)\frac{\hat{R}(t_<)}{R^2(t_<)}+\frac{1}{c^2}\frac{d^2}{dt^2}\hat{R}(t_<)\right] \tag{4}$$

Note, all quantities on the RHS of (4) are evaluated at the earlier time $t_<$ *before* the t derivatives are taken. Find an expression for the corresponding $\vec{B}(\vec{r},t)$. [Hint: Is it true that $c\vec{B}(\vec{r},t) = \hat{R}(t_<)\times\vec{E}(\vec{r},t)$?]

[7] As discussed in lecture, a time-dependent point electric dipole at the origin, with $\vec{p}(t) = p(t)\ \hat{z}$, produces radiation whose electric field in cylindrical coordinates is of the form

$$\vec{E}(\rho,z,t) = \frac{1}{4\pi\varepsilon_0\rho}\left(\frac{dR}{dz}\,\hat{\rho}-\frac{dR}{d\rho}\,\hat{z}\right)\ ,\quad \text{where}\quad R(\rho,z,t) = \rho\,\frac{\partial}{\partial\rho}\left(\frac{p(t-r/c)}{r}\right)\ , \tag{5}$$

and of course, $r^2 = \rho^2 + z^2$. Plot some field lines in the ρz-plane for various times, when $p(t) = p_0\cos(\omega t)$. Are these field lines given by curves for which $R(\rho,z,t) = K$, for various constant K? Show that points where the electric field can vanish, and therefore produce intersecting field lines, are given by solutions of the equations

$$z = 0 \quad \text{and} \quad \frac{d^2p(t-r/c)}{dt^2} = c^2K/\rho \tag{6}$$

[8] Compute $\vec{A}_{\text{rad}}$, $\vec{E}_{\text{rad}}$, $\vec{B}_{\text{rad}}$ and $\left\langle\frac{d\mathcal{P}}{d\Omega}\right\rangle$ for an ideal, harmonic, "sine-wave" antenna defined by

$$\vec{J}(\vec{r},t) = I\ \hat{z}\ \cos(\omega t)\ \ \Theta(d-|z|)\ \ \delta(x)\ \ \delta(y)\ \ \sin k(d-|z|) \tag{7}$$

Here $\omega = kc$, the δs are Dirac deltas, and Θ is the Helmholtz step function.

[9] A uniformly charged straight-line segment of length L is rotating at constant angular frequency ω about an axis through its center and perpendicular to the segment. Compute $\vec{E}$ and $\vec{B}$ for all points *on the axis of rotation.* [Hint: Feynman's expression (4) for the field of a moving point charge is especially useful when R is constant in time.]

[10] Have some fun with vector spherical harmonics! Compute the two integrals:

$$\int\left(\vec{Y}_{l'm'}(\hat{r})\right)^*\cdot\left(\vec{r}\times\vec{Y}_{lm}(\hat{r})\right)d\Omega\ ,\qquad \int\left(\vec{Y}_{l'm'}(\hat{r})\right)^*\cdot\left(\vec{\nabla}\times\vec{Y}_{lm}(\hat{r})\right)d\Omega\ . \tag{8}$$

Also show that

$$\left|\vec{Y}_{1,\pm1}\right|^2 = \left|\hat{r}\times\vec{Y}_{1,\pm1}\right|^2 = \frac{3}{16\pi}\left(1+\cos^2\theta\right)\ , \tag{9}$$

$$\left(\hat{r}\times\vec{Y}_{1,\pm1}\right)^*\cdot\vec{Y}_{1,\pm1} = \hat{r}\cdot\left(\vec{Y}^*_{1,\pm1}\times\vec{Y}_{1,\pm1}\right) = \pm\frac{3i}{8\pi}\cos\theta\ . \tag{10}$$

[11] Establish the "near-field" behavior of TM fields. Show that

$$\vec{E}(\vec{r},t)\underset{kr\ll1}{\sim} -Z_0\,a^E_{lm}\,\vec{\nabla}\left(\frac{Y_{lm}}{r^{l+1}}\right)\ ,\qquad \vec{H}(\vec{r},t)\underset{kr\ll1}{\sim} -\frac{k}{l}\,a^E_{lm}\,\frac{\overrightarrow{Y_{lm}}}{r^{l+1}}\ . \tag{11}$$

Hence the alternative name "electric multipole" for TM fields, because in this case $\overrightarrow{E}(\overrightarrow{r},t) \underset{kr\ll 1}{\sim} -\overrightarrow{\nabla}\Phi^E_{lm}$ where $\Phi^E_{lm} \equiv Z_0\, a^E_{lm}\, \frac{Y_{lm}}{r^{l+1}}$. Note that $\left|\overrightarrow{H}(\overrightarrow{r},t)\right| \underset{kr\ll 1}{\sim} kr\left|\overrightarrow{E}(\overrightarrow{r},t)\right|$. Also note that TE (i.e. "magnetic multipole") fields can be obtained from the TM fields by the "duality" transformation $\left(\overrightarrow{E},\overrightarrow{H}\right) \longrightarrow \left(-Z_0\overrightarrow{H},\overrightarrow{E}/Z_0\right)$ where $Z_0 \equiv \sqrt{\mu_0/\varepsilon_0} \approx 377\ \Omega$.

[12] For classical electrodynamics, it is useful to define the angular momentum differential operator as $\overrightarrow{L} \equiv -i\overrightarrow{r}\times\overrightarrow{\nabla}$. When $\rho = 0 = \overrightarrow{J}$, show that *all* of the Maxwell equations are satisfied by

$$\begin{aligned} \overrightarrow{E}(\overrightarrow{r},t) &= \left(\overrightarrow{\nabla}\times\overrightarrow{L}\right)\Omega^E(\overrightarrow{r},t) - \frac{1}{c}\overrightarrow{L}\ \partial_t\Omega^M(\overrightarrow{r},t) \qquad (12)\\ c\overrightarrow{B}(\overrightarrow{r},t) &= \left(\overrightarrow{\nabla}\times\overrightarrow{L}\right)\Omega^M(\overrightarrow{r},t) + \frac{1}{c}\overrightarrow{L}\ \partial_t\Omega^E(\overrightarrow{r},t) \qquad (13)\end{aligned}$$

where now $\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\Omega^{E,M}(\overrightarrow{r},t) = 0$. In terms of these "Debye potentials" $\Omega^E(\overrightarrow{r},t)$ and $\Omega^M(\overrightarrow{r},t)$, find more conventional scalar and vector potentials, $\Phi(\overrightarrow{r},t)$ and $\overrightarrow{A}(\overrightarrow{r},t)$, so that the electric and magnetic fields can be written in the usual form as $\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \partial_t\overrightarrow{A}$ and $\overrightarrow{B} = \overrightarrow{\nabla}\times\overrightarrow{A}$. (Do not be alarmed if your answers for Φ and $\overrightarrow{A}$ involve $i\Omega^E$ and $i\Omega^M$!) Does your result for Φ permit you to have a $1/r$ Coulomb potential?

[13] As you may recall from electrostatics, the electric dipole moment induced in a perfectly conducting sphere of radius R when placed in a uniform electric field $\overrightarrow{E}$ is $\overrightarrow{p} = 4\pi\varepsilon_0 R^3\overrightarrow{E}$. Similarly, the magnetic dipole moment induced in a perfectly conducting sphere by a uniform magnetic field $\overrightarrow{B}$ is $\overrightarrow{m} = -2\pi R^3\overrightarrow{B}/\mu_0$. Use these results and your classroom notes to obtain the differential and total scattering cross sections for unpolarized, monochromatic E&M plane waves, scattering from a perfectly conducting sphere of radius R, in the low frequency limit $kR \ll 1$, namely,

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= k^4R^6\left(\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta\right) \qquad (14)\\ \sigma &= \int\left(\frac{d\sigma}{d\Omega}\right)d\Omega = \frac{10\pi}{3}\ k^4R^6\ . \qquad (15)\end{aligned}$$

Here, θ is the angle between the incident wave vector, $\overrightarrow{k}_{\text{in}} = k\,\widehat{z}$, and the scattered wave vector, $\overrightarrow{k} = k\,\widehat{r}$.

[14] For another look at the previous problem, use the results from Problem [10] to verify that scattering *either* right- or left-circularly polarized, monochromatic, E&M plane waves from a perfectly conducting sphere of radius R, in the low frequency limit $kR \ll 1$, results in the same differential and total cross-sections as in (14) and (15).

[15] Use the exact boundary conditions for a monochromatic plane E&M wave incident on a dielectric sphere to deduce the coefficients that appear in the spherical wave expansions of the scattered electric and magnetic fields (i.e. the Mie scattering results as discussed in class).

[16] An object scatters an incident, monochromatic plane E&M wave with wave vector $\overrightarrow{k}_{\text{in}} = k\,\widehat{z}$. Use the Maxwell stress tensor to show the time-averaged force on the object can be written in terms of the total and differential scattering cross-sections as

$$\left\langle\overrightarrow{F}\right\rangle = \frac{1}{c}\ I_{\text{in}}\left(\sigma_{\text{total}}\overrightarrow{k}_{\text{in}} - \int\widehat{r}\ \frac{d\sigma}{d\Omega}\ d\Omega\right) \qquad (16)$$

where $I_{\text{in}} = \left|\left\langle\overrightarrow{S}\right\rangle\right|$ is the "intensity" of the incident wave.

[17] Lorentz transform

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \qquad (17)$$

from the rest frame of a charged point particle to a frame where the particle has velocity $\overrightarrow{v} = v\,\widehat{z}$ with constant speed v.

[18] Lorentz transform Φ for a uniformly charged straight-line segment of length L from the segment's rest frame to moving frames corresponding to two cases: [a] the segment moves with constant velocity $\overrightarrow{v}$ along the axis of the segment, and [b] the segment moves with constant velocity $\overrightarrow{v}$ perpendicular to the axis of the segment. (It may help to suppose the center of the segment is initially at a common origin for all the inertial frames.) What is the transformed vector potential $\overrightarrow{A}$ for these two cases?

[19] Compute $\overrightarrow{E}$ and $\overrightarrow{B}$ for the moving line segments (i.e. both cases) of the previous problem. For $\overrightarrow{v} = v\,\widehat{z}$ with the center of the segment initially at $\overrightarrow{r} = 0$, plot E_x, E_z, and B_y versus t, at the location $(x, y, z) = (L, 0, 0)$, for various speeds (e.g. $v = \frac{1}{2}c$, $\frac{3}{4}c$, $\frac{7}{8}c$, $\frac{15}{16}c$, etc.). Note for plotting purposes there are two independent situations here for the case when the segment is moving perpendicular to its own axis, namely, with segment orientation given by $\overrightarrow{L} = L\,\widehat{x}$ and $\overrightarrow{L} = L\,\widehat{y}$.

[20] The covariant form of the Lienard-Wiechert potential for a point charge is

$$A^\mu = \frac{q}{4\pi\varepsilon_0}\,\frac{v^\mu}{cR^\alpha v_\alpha}\bigg|_{R^\beta R_\beta = 0 \text{ and } R^0 > 0} \tag{18}$$

where $R^\alpha = r^\alpha - \mathfrak{r}^\alpha$ is the four-vector from the particle's location, as defined by the trajectory $\mathfrak{r}^\alpha(\tau)$, to the field (observation) point, where the particle's four-velocity is $v^\mu = d\mathfrak{r}^\mu/d\tau$, and where $d\tau = \sqrt{1 - \overrightarrow{v}\cdot\overrightarrow{v}/c^2}\,dt$ is the Lorentz invariant "proper time" increment for the particle. Show that the causal field strength for the particle is given by

$$F^{\mu\nu} = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{cR^\alpha v_\alpha}\,\frac{d}{d\tau}\left(\frac{R^\mu v^\nu - R^\nu v^\mu}{R^\alpha v_\alpha}\right)\bigg|_{R^\beta R_\beta = 0 \text{ and } R^0 > 0} \tag{19}$$

[21] Consider the relativistic particle Lorentz invariant $a^\mu a_\mu$, where in terms of proper time,

$$a^\mu = \frac{dv^\mu}{d\tau} = \frac{d^2 x^\mu}{d\tau^2} \tag{20}$$

Express this invariant in terms of the particle's 3-velocity and 3-acceleration, i.e. show that

$$a^\mu a_\mu = \gamma^6\left(\left|\overrightarrow{a}\right|^2 - \left|\overrightarrow{\beta}\times\overrightarrow{a}\right|^2\right) \tag{21}$$