PHY753, Spring 2022, Assigned Problems

For clarification or to point out a typo (or worse!) please send email to curtright@miami.edu

[0] Find the URL for the course webpage and email it to *curtright@miami.edu* (just the URL, please).

[1] For a given "source" $S(x,t)$ in one space dimension, with $-\infty \leq x \leq +\infty$ and $-\infty \leq t \leq +\infty$, solve

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right) F(x,t) = S(x,t)$$

for the "field" $F(x,t)$ as an integral involving a causal Green function $G(x,t;x',t')$. Express the Green function in closed form in terms of a Heaviside step function.

[2] Simplify the Schott-Panofsky-Phillips-Jefimenko expression for the causal electric field $\overrightarrow{E}(\overrightarrow{r},t)$ when the current density has linear dependence and the charge density has quadratic dependence on t. That is to say, let

$$\overrightarrow{J}(\overrightarrow{r},t) = \overrightarrow{J}(\overrightarrow{r}) + t\,\overrightarrow{K}(\overrightarrow{r}) \ , \quad \rho(\overrightarrow{r},t) = \rho(\overrightarrow{r}) + t\,\varrho(\overrightarrow{r}) + \frac{1}{2}\,t^2\,\zeta(\overrightarrow{r})$$

Compare $\overrightarrow{E}(\overrightarrow{r},t)$ to the "instantaneous Coulomb field" as given by

$$\overrightarrow{E}_{Coulomb}(\overrightarrow{r},t) = \frac{1}{4\pi\varepsilon_0}\int d^3r'\ \rho(\overrightarrow{r}',t)\,\frac{\widehat{R}}{R^2}$$

where $\overrightarrow{R} = \overrightarrow{r} - \overrightarrow{r}'$ and the charge density is evaluated at the field time t, *not* the causal time $t_< = t - R/c$.

[3] Current $I(t)$ flows along an ideal line (e.g. the negative z axis, with $I(t) > 0$ for current flowing towards $z = 0$) and terminates at a point (e.g. the origin of coordinates) where it accumulates to produce charge $Q(t)$. Assume that charge is *not* accumulated at any *other* point. (a) Find the relation in general between $I(t)$ and $Q(t)$, and in particular find $Q(t)$ when the current has linear time dependence, $I(t) = I_0 + K_0 t$ where I_0 and K_0 are constants. (b) Write expressions for $\rho(\overrightarrow{r},t)$ and $\overrightarrow{J}(\overrightarrow{r},t)$ making use of Dirac deltas and the Heaviside step function Θ, both for general $I(t)$ & $Q(t)$ and for the particular case when the current is linear in time. (c) For such an ideal current flow, simplify the integrals in the Schott–Panofsky-Phillips–Jefimenko expressions as much as possible, and explicitly evaluate the integrals when the current is linear in time to obtain $\overrightarrow{B}(\overrightarrow{r},t)$ and $\overrightarrow{E}(\overrightarrow{r},t)$ for all points that avoid the current and charge. If you encounter a divergent integral, do something to fix it!

[4] Show for a uniformly moving point charge that $\overrightarrow{E}$ and $\overrightarrow{B}$ obey *all* of Maxwell's equations, where $\rho(\overrightarrow{r},t) = q\,\delta^3(\overrightarrow{r} - \overrightarrow{v}t)$ and $\overrightarrow{J}(\overrightarrow{r},t) = \overrightarrow{v}\,\rho(\overrightarrow{r},t)$. Note that the coordinates in these charge and current densities are those of the observer for whom the charge's trajectory is $\overrightarrow{\mathfrak{r}}(t) = \overrightarrow{v}t$.

[5] For any given *localized* charge and current densities, it is possible to define Φ causally with respect to *any* v of your choosing!!! Namely,

$$\Phi_{v\text{-gauge}}(\overrightarrow{r},t) = \frac{1}{4\pi\varepsilon_0}\int \frac{1}{|\overrightarrow{r} - \overrightarrow{s}|}\ \rho\left(\overrightarrow{s}, t - \frac{1}{v}|\overrightarrow{r} - \overrightarrow{s}|\right) d^3s \tag{1}$$

where the "v-dependent retarded time" at the source point is now $t - |\overrightarrow{r} - \overrightarrow{s}|/v$. What inhomogeneous, second-order, partial differential equation does $\Phi_{v\text{-gauge}}$ obey? Construct a gauge transformation $\chi(\overrightarrow{r},t)$ that connects this v-gauge to the Lorenz gauge. Work out an explicit expression for the vector potential $\overrightarrow{A}_{v\text{-gauge}}(\overrightarrow{r},t)$ in this class of gauges. What inhomogeneous, second-order, partial differential equation does $\overrightarrow{A}_{v\text{-gauge}}$ obey? Show that the v-gauge potentials satisfy the constraint

$$\overrightarrow{\nabla}\cdot\overrightarrow{A}_{v\text{-gauge}} + \frac{1}{v^2}\,\partial_t\Phi_{v\text{-gauge}} = 0 \tag{2}$$

and take note of the Coulomb ($v \to \infty$) and Lorenz gauge ($v \to c$) limits of all your results.

[6] From the Lienard-Wiechert potentials Φ and $\vec{A}$ for a moving point particle, show that the electric field is given by:

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{R^2}\,\frac{1}{\left(1-\hat{R}\cdot\vec{v}/c\right)^3}\left[\left(1-v^2/c^2\right)\left(\hat{R}-\vec{v}/c\right)+\frac{1}{c^2}\vec{R}\times\left(\left(\hat{R}-\vec{v}/c\right)\times\vec{a}\right)\right]\Bigg|_{t_<} \tag{3}$$

where the particle's trajectory determines $\vec{\mathfrak{r}}(t)$, $\vec{v}(t) = d\vec{\mathfrak{r}}(t)/dt$, and $\vec{a}(t) = d^2\vec{\mathfrak{r}}(t)/dt^2$, where $\vec{R}(t) = \vec{r} - \vec{\mathfrak{r}}(t)$, and where *all* quantities on the RHS of (3) are evaluated at the earlier time $t_<$ defined implicitly by $t_< = t - R(t_<)/c$. Show equivalence between (3) and the Feynman expression for the electric field of a moving point charge, namely,

$$\vec{E}(\vec{r},t) = \frac{q}{4\pi\varepsilon_0}\left[\left(1+\frac{R(t_<)}{c}\frac{d}{dt}\right)\frac{\hat{R}(t_<)}{R^2(t_<)}+\frac{1}{c^2}\frac{d^2}{dt^2}\hat{R}(t_<)\right] \tag{4}$$

Note, all quantities on the RHS of (4) are evaluated at the earlier time $t_<$ *before* the t derivatives are taken. Find an expression for the corresponding $\vec{B}(\vec{r},t)$. [Hint: Is it true that $c\vec{B}(\vec{r},t) = \hat{R}(t_<)\times\vec{E}(\vec{r},t)$?]

[7] As discussed in lecture, a time-dependent point electric dipole at the origin, with $\vec{p}(t) = p(t)\;\hat{z}$, produces radiation whose electric field in cylindrical coordinates is of the form

$$\vec{E}(\rho,z,t) = \frac{1}{4\pi\varepsilon_0\rho}\left(\frac{dR}{dz}\;\hat{\rho}-\frac{dR}{d\rho}\;\hat{z}\right)\;,\quad\text{where}\quad R(\rho,z,t) = \rho\,\frac{\partial}{\partial\rho}\left(\frac{p(t-r/c)}{r}\right)\;, \tag{5}$$

and of course, $r^2 = \rho^2 + z^2$. Plot some field lines in the ρz-plane for various times, when $p(t) = p_0\cos(\omega t)$. Are these field lines given by curves for which $R(\rho,z,t) = K$, for various constant K? Show that points where the electric field can vanish, and therefore produce intersecting field lines, are given by solutions of the equations

$$z = 0 \quad\text{and}\quad \frac{d^2p(t-r/c)}{dt^2} = c^2K/\rho \tag{6}$$

[8] Compute $\vec{A}_{\text{rad}}$, $\vec{E}_{\text{rad}}$, $\vec{B}_{\text{rad}}$ and $\langle\frac{d\mathcal{P}}{d\Omega}\rangle$ for an ideal, harmonic, "sine-wave" antenna defined by

$$\vec{J}(\vec{r},t) = I\;\hat{z}\;\cos(\omega t)\;\;\Theta(d-|z|)\;\;\delta(x)\;\;\delta(y)\;\;\sin k(d-|z|) \tag{7}$$

Here $\omega = kc$, the δs are Dirac deltas, and Θ is the Heaviside step function.

[9] A uniformly charged straight-line segment of length L is rotating at constant angular frequency ω about an axis through its center and perpendicular to the segment. Compute $\vec{E}$ and $\vec{B}$ for all points *on the axis of rotation*. [Hint: Feynman's expression (4) for the field of a moving point charge is especially useful when R is constant in time.]

[10] Have some fun with vector spherical harmonics! Compute the two integrals:

$$\int\left(\vec{Y}_{l'm'}(\hat{r})\right)^*\cdot\left(\vec{r}\times\vec{Y}_{lm}(\hat{r})\right)d\Omega\;,\qquad\int\left(\vec{Y}_{l'm'}(\hat{r})\right)^*\cdot\left(\vec{\nabla}\times\vec{Y}_{lm}(\hat{r})\right)d\Omega\;. \tag{8}$$

Also show that

$$\left|\vec{Y}_{1,\pm1}\right|^2 = \left|\hat{r}\times\vec{Y}_{1,\pm1}\right|^2 = \frac{3}{16\pi}\left(1+\cos^2\theta\right)\;, \tag{9}$$

$$\left(\hat{r}\times\vec{Y}_{1,\pm1}\right)^*\cdot\vec{Y}_{1,\pm1} = \hat{r}\cdot\left(\vec{Y}^*_{1,\pm1}\times\vec{Y}_{1,\pm1}\right) = \pm\frac{3i}{8\pi}\cos\theta\;. \tag{10}$$

[11] Establish the "near-field" behavior of TM fields. Show that

$$\vec{E}(\vec{r},t)\underset{kr\ll1}{\sim} -Z_0\;a^E_{lm}\,\vec{\nabla}\left(\frac{Y_{lm}}{r^{l+1}}\right)\;,\qquad\vec{H}(\vec{r},t)\underset{kr\ll1}{\sim} -\frac{k}{l}\;a^E_{lm}\,\frac{\overrightarrow{Y_{lm}}}{r^{l+1}}\;. \tag{11}$$

Hence the alternative name "electric multipole" for TM fields, because in this case $\overrightarrow{E}(\overrightarrow{r},t) \underset{kr\ll 1}{\sim} -\overrightarrow{\nabla}\Phi^E_{lm}$ where $\Phi^E_{lm} \equiv Z_0\ a^E_{lm}\ \frac{Y_{lm}}{r^{l+1}}$. Note that $\left|\overrightarrow{H}(\overrightarrow{r},t)\right| \underset{kr\ll 1}{\sim} kr\left|\overrightarrow{E}(\overrightarrow{r},t)\right|$. Also note that TE (i.e. "magnetic multipole") fields can be obtained from the TM fields by the "duality" transformation $\left(\overrightarrow{E},\overrightarrow{H}\right) \longrightarrow \left(-Z_0\overrightarrow{H},\overrightarrow{E}/Z_0\right)$ where $Z_0 \equiv \sqrt{\mu_0/\varepsilon_0} \approx 377\ \Omega$.

[12] For classical electrodynamics, it is useful to define the angular momentum differential operator as $\overrightarrow{L} \equiv -i\overrightarrow{r}\times\overrightarrow{\nabla}$. When $\rho = 0 = \overrightarrow{J}$, show that *all* of the Maxwell equations are satisfied by

$$\overrightarrow{E}(\overrightarrow{r},t) = \left(\overrightarrow{\nabla}\times\overrightarrow{L}\right)\Omega^E(\overrightarrow{r},t) - \frac{1}{c}\overrightarrow{L}\ \partial_t\Omega^M(\overrightarrow{r},t) \tag{12}$$

$$c\overrightarrow{B}(\overrightarrow{r},t) = \left(\overrightarrow{\nabla}\times\overrightarrow{L}\right)\Omega^M(\overrightarrow{r},t) + \frac{1}{c}\overrightarrow{L}\ \partial_t\Omega^E(\overrightarrow{r},t) \tag{13}$$

where now $\left(\frac{1}{c^2}\partial_t^2 - \nabla^2\right)\Omega^{E,M}(\overrightarrow{r},t) = 0$. In terms of these "Debye potentials" $\Omega^E(\overrightarrow{r},t)$ and $\Omega^M(\overrightarrow{r},t)$, find more conventional scalar and vector potentials, $\Phi(\overrightarrow{r},t)$ and $\overrightarrow{A}(\overrightarrow{r},t)$, so that the electric and magnetic fields can be written in the usual form as $\overrightarrow{E} = -\overrightarrow{\nabla}\Phi - \partial_t\overrightarrow{A}$ and $\overrightarrow{B} = \overrightarrow{\nabla}\times\overrightarrow{A}$. (Do not be alarmed if your answers for Φ and $\overrightarrow{A}$ involve $i\Omega^E$ and $i\Omega^M$!) Does your result for Φ permit you to have a $1/r$ Coulomb potential?

[13] As you may recall from electrostatics, the electric dipole moment induced in a perfectly conducting sphere of radius R when placed in a uniform electric field $\overrightarrow{E}$ is $\overrightarrow{p} = 4\pi\varepsilon_0 R^3\overrightarrow{E}$. Similarly, the magnetic dipole moment induced in a perfectly conducting sphere by a uniform magnetic field $\overrightarrow{B}$ is $\overrightarrow{m} = -2\pi R^3\overrightarrow{B}/\mu_0$. Use these results and your classroom notes to obtain the differential and total scattering cross sections for unpolarized, monochromatic E&M plane waves, scattering from a perfectly conducting sphere of radius R, in the low frequency limit $kR \ll 1$, namely,

$$\frac{d\sigma}{d\Omega} = k^4R^6\left(\frac{5}{8}\left(1+\cos^2\theta\right) - \cos\theta\right) \tag{14}$$

$$\sigma = \int\left(\frac{d\sigma}{d\Omega}\right)d\Omega = \frac{10\pi}{3}\ k^4R^6\ . \tag{15}$$

Here, θ is the angle between the incident wave vector, $\overrightarrow{k}_{\text{in}} = k\ \widehat{z}$, and the scattered wave vector, $\overrightarrow{k} = k\ \widehat{r}$.

[14] For another look at the previous problem, use the results from Problem [10] to verify that scattering *either* right- or left-circularly polarized, monochromatic, E&M plane waves from a perfectly conducting sphere of radius R, in the low frequency limit $kR \ll 1$, results in the same differential and total cross-sections as in (14) and (15).

[15] Use the exact boundary conditions for a monochromatic plane E&M wave incident on a dielectric sphere to deduce the coefficients that appear in the spherical wave expansions of the scattered electric and magnetic fields (i.e. the Mie scattering results as discussed in class).

[16] An object scatters an incident, monochromatic plane E&M wave with wave vector $\overrightarrow{k}_{\text{in}} = k\ \widehat{z}$. Use the Maxwell stress tensor to show the time-averaged force on the object can be written in terms of the total and differential scattering cross-sections as

$$\left\langle\overrightarrow{F}\right\rangle = \frac{1}{c}\ I_{\text{in}}\left(\sigma_{\text{total}}\overrightarrow{k}_{\text{in}} - \int\widehat{r}\ \frac{d\sigma}{d\Omega}\ d\Omega\right) \tag{16}$$

where $I_{\text{in}} = \left|\left\langle\overrightarrow{S}\right\rangle\right|$ is the "intensity" of the incident wave.

[17] Lorentz transform

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix} \tag{17}$$

from the rest frame of a charged point particle to a frame where the particle has velocity $\vec{v} = v\,\hat{z}$ with constant speed v.

[18] Lorentz transform Φ for a uniformly charged straight-line segment of length L from the segment's rest frame to moving frames corresponding to two cases: [a] the segment moves with constant velocity $\vec{v}$ along the axis of the segment, and [b] the segment moves with constant velocity $\vec{v}$ perpendicular to the axis of the segment. (It may help to suppose the center of the segment is initially at a common origin for all the inertial frames.) What is the transformed vector potential $\vec{A}$ for these two cases?

[19] Compute $\vec{E}$ and $\vec{B}$ for the moving line segments (i.e. both cases) of the previous problem. For $\vec{v} = v\,\hat{z}$ with the center of the segment initially at $\vec{r} = 0$, plot E_x, E_z, and B_y versus t, at the location $(x, y, z) = (L, 0, 0)$, for various speeds (e.g. $v = \frac{1}{2}c,\ \frac{3}{4}c,\ \frac{7}{8}c,\ \frac{15}{16}c$, etc.). Note for plotting purposes there are two independent situations here for the case when the segment is moving perpendicular to its own axis, namely, with segment orientation given by $\vec{L} = L\,\hat{x}$ and $\vec{L} = L\,\hat{y}$.

[20] The covariant form of the Lienard-Wiechert potential for a point charge is

$$A^{\mu} = \frac{q}{4\pi\varepsilon_0}\,\frac{v^{\mu}}{cR^{\alpha}v_{\alpha}}\bigg|_{R^{\beta}R_{\beta}=0 \text{ and } R^0>0} \tag{18}$$

where $R^{\alpha} = r^{\alpha} - \mathfrak{r}^{\alpha}$ is the four-vector from the particle's location, as defined by the trajectory $\mathfrak{r}^{\alpha}(\tau)$, to the field (observation) point, where the particle's four-velocity is $v^{\mu} = d\mathfrak{r}^{\mu}/d\tau$, and where $d\tau = \sqrt{1 - \vec{v}\cdot\vec{v}/c^2}\,dt$ is the Lorentz invariant "proper time" increment for the particle. Show that the causal field strength for the particle is given by

$$F^{\mu\nu} = \frac{q}{4\pi\varepsilon_0}\,\frac{1}{cR^{\alpha}v_{\alpha}}\,\frac{d}{d\tau}\left(\frac{R^{\mu}v^{\nu} - R^{\nu}v^{\mu}}{R^{\alpha}v_{\alpha}}\right)\bigg|_{R^{\beta}R_{\beta}=0 \text{ and } R^0>0} \tag{19}$$

[21] Consider the relativistic particle Lorentz invariant $a^{\mu}a_{\mu}$, where in terms of proper time,

$$a^{\mu} = \frac{dv^{\mu}}{d\tau} = \frac{d^2x^{\mu}}{d\tau^2} \tag{20}$$

Express this invariant in terms of the particle's 3-velocity and 3-acceleration, i.e. show that

$$a^{\mu}a_{\mu} = \gamma^6\left(|\vec{a}|^2 - \left|\vec{\beta}\times\vec{a}\right|^2\right) \tag{21}$$