Functional roots of the sine function

Notes by Thomas Curtright, February 2011.

We construct these numerically, first as series,[1] and then we improve on the numerical approximations by exploiting the functional relations that *should* be obeyed by the exact functions. For example,

$$\mathrm{rin}\,(x) = \mathrm{arcrin}\,(\sin\,(x)) \ , \quad \mathrm{qin}\,(x) = \mathrm{arcqin}\,(\mathrm{arcrin}\,(\sin\,(x))) \ , \tag{1}$$

etc., where rin is the functional square-root of sin, qin is the functional fourth-root, and arcrin and arcqin are (the principal branches of) their respective inverses. That is to say, $\sin\,(x) = \mathrm{rin}\,(\mathrm{rin}\,(x))$, $\mathrm{rin}\,(x) = \mathrm{qin}\,(\mathrm{qin}\,(x))$, $\mathrm{arcrin}\,(\mathrm{rin}\,(x)) = x = \mathrm{arcqin}\,(\mathrm{qin}\,(x))$, etc.

As a general rule, it is more accurate to construct the functional positive roots by acting on sin with the series approximations for the inverse roots (i.e. the functional negative roots) since the series seem to be reasonably good numerical approximations for $0 \leq x \approx 1$, but visibly fail to be accurate as x approaches $\pi/2$.

In fact, an elementary extension of this idea would seem to allow numerical computation of the various functional roots to arbitrary accuracy. The conjectured theorem is

$$\sin_{[f]}\,(x) = \lim_{n\to\infty} \sin_{[-n]}\left(\sin_{[f]}^{\text{series}}\left(\sin_{[n]}\,(x)\right)\right) \tag{2}$$

where $\sin_{[f]}^{\text{series}}$ is any conveniently truncated asymptotic series approximation to $\sin_{[f]}$, and where

$$\begin{aligned} \sin_{[n+1]}\,(x) &= \sin\left(\sin_{[n]}\,(x)\right) \ , \quad \sin_{[1]}\,(x) = \sin\,(x) \\ \sin_{[-n-1]}\,(x) &= \arcsin\left(\sin_{[-n]}\,(x)\right) \ , \quad \sin_{[-1]}\,(x) = \arcsin\,(x) \ . \end{aligned} \tag{3}$$

In particular,

$$\mathrm{rin}\,(x) = \lim_{n\to\infty} \sin_{[-n]}\left(\mathrm{rin}_{\text{series}}\left(\sin_{[n]}\,(x)\right)\right) \tag{4a}$$

$$\mathrm{qin}\,(x) = \lim_{n\to\infty} \sin_{[-n]}\left(\mathrm{qin}_{\text{series}}\left(\sin_{[n]}\,(x)\right)\right) \tag{4b}$$

etc., where $\mathrm{rin}_{\text{series}}$ and $\mathrm{qin}_{\text{series}}$ are given below. Rather than taking the limit, of course, progressively more accurate numerical results follow just by taking successively larger n on the RHS of (2).

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[1] Except for integer functional iterates of $\sin x$, which of course are entire functions, we expect these series to have at best *finite* radii of convergence [1]. The actual series data for rin and qin suggests the series are in fact asymptotic, and not convergent. However, for purposes of constructing the graphs to follow, asymptotic series suffice.

[1] P. Erdös and E. Jabotinsky "On Analytic Iteration" Journal D'Analyse Mathematique **8** (1960) 361-376.

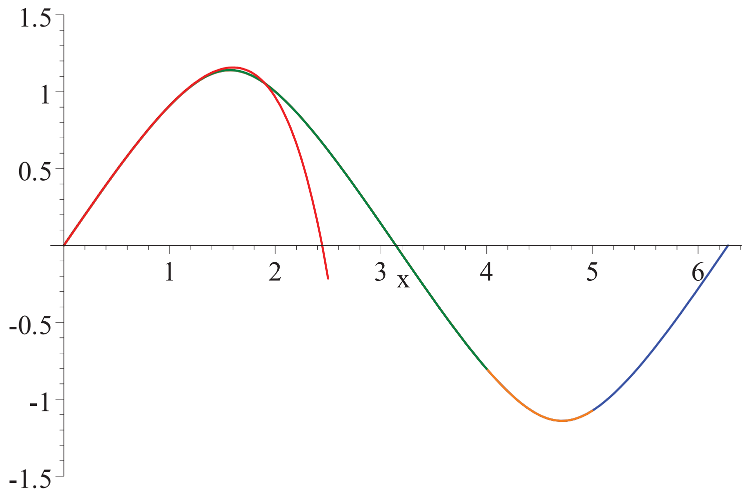
To emphasize the point made in (2), we take for rin the simple yet rather crude seventh order series

$$r(x) = x - \frac{1}{12}x^3 - \frac{1}{160}x^5 - \frac{53}{40\,320}x^7 \tag{5}$$

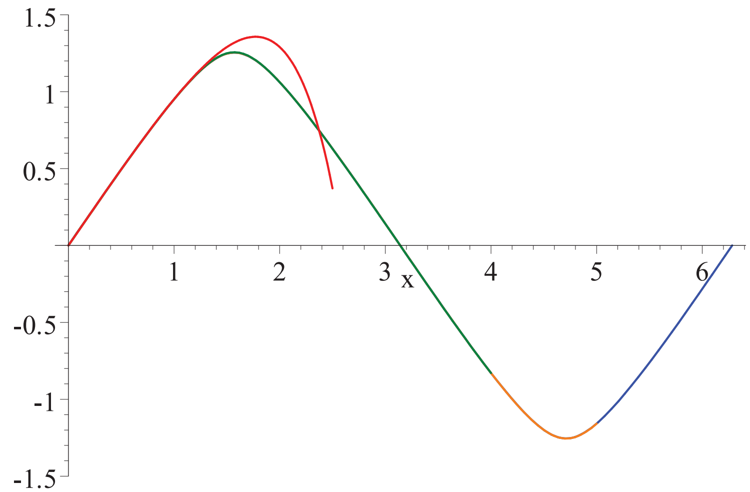
and forge it into an *excellent* numerical approximation just by functional conjugation with sin and arcsin. We plot the 7th order series and compare it to (4a) for $n = 1, 2, 3$, and 4. While the series itself is lousy, for $x \approx \pi/2$ and above, just the first arcsin − sin conjugation works very well for a complete cycle $0 \leq x \leq 2\pi$. Double and triple conjugations give better approximations, but any changes in the first graphs below are only barely discernible. Similarly we plot the seventh order series approximation for qin

$$q(x) = x - \frac{1}{24}x^3 - \frac{11}{1920}x^5 - \frac{97}{64\,512}x^7 \tag{6}$$

and compare it to (4b) for $n = 1, 2, 3$, and 4.



$r(x)$ and its conjugations.

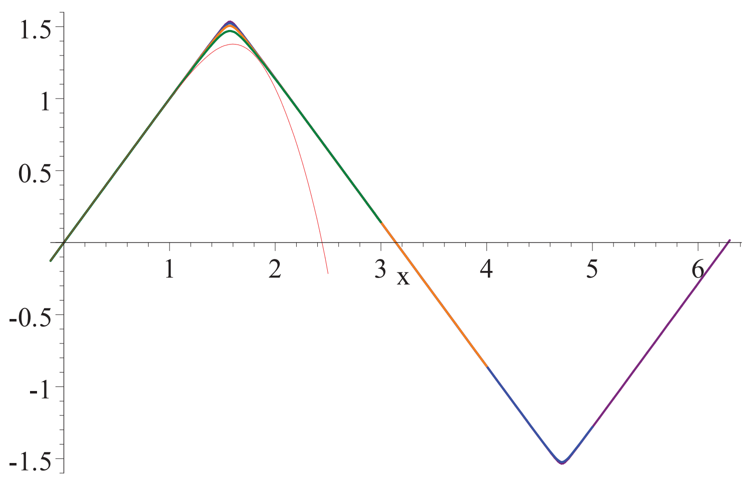


$q(x)$ and its conjugations.

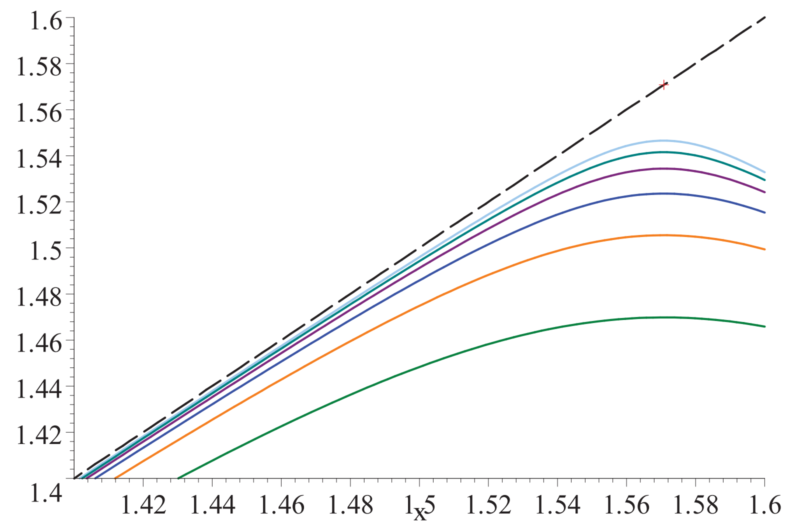
Now some acid tests. Does this procedure really shrink the error without magnifying it again in the end? Consider conjugations of the approximate inverse composed with the approximate function. For arcrin we again take a crude 7th order series approximation.

$$a(x) = x + \frac{1}{12}x^3 + \frac{13}{480}x^5 + \frac{167}{13\,440}x^7$$

Then $a(r(x)) = x - \frac{18\,377}{2903\,040}x^9 + O\left(x^{11}\right) = x - 6.3\times 10^{-3}x^9 + O\left(x^{11}\right)$ — not exactly the identity map — so there is indeed a *relative* error with a maximum value on the quarter cycle of $6.3\times10^{-3}\left(\frac{\pi}{2}\right)^9/\frac{\pi}{2} = 0.23$. But, upon conjugating with sin and arcsin, voila!



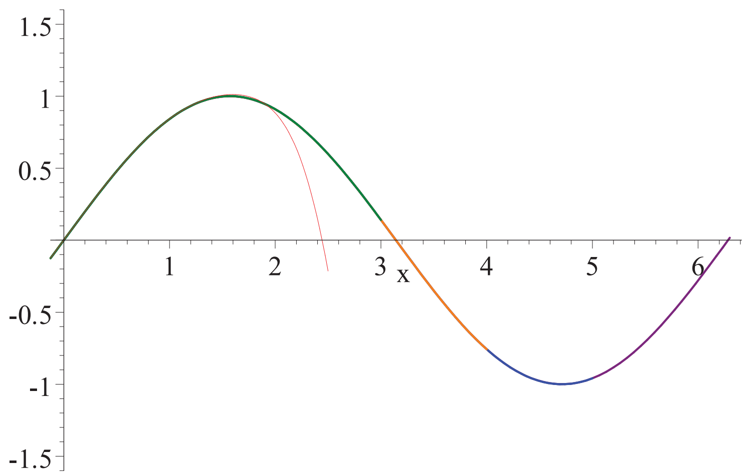
$a(r(x))$ and its conjugations.



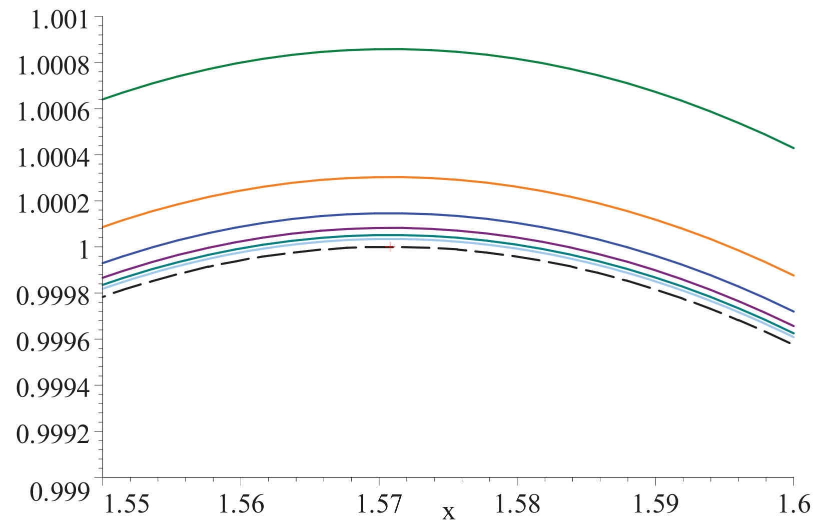
$a(r(x))$ and its conjugations $\sin_{[-n]}\left(a\left(r\left(\sin_{[n]}(x)\right)\right)\right)$ for $n = 1, 2, 3, 4, 5, 6$. The red cross is the point $(\pi/2, \pi/2)$.

The error is indeed reduced. The maximum relative error here for the six-fold conjugated 7th order series is about 1×10^{-2}.

Similarly, we compare $r\left(r\left(x\right)\right)$ and its conjugations to $\sin\left(x\right)$.



$r\left(r\left(x\right)\right)$ and its conjugations $\sin_{[-n]}\left(r\left(r\left(\sin_{[n]}\left(x\right)\right)\right)\right)$ for $n = 1, 2, 3, 4$.



The conjugations $\sin_{[-n]}\left(r\left(r\left(\sin_{[n]}\left(x\right)\right)\right)\right)$ for $n = 1, 2, 3, 4, 5, 6$ ($n = 1$ in green) compared to $\sin x$ (black dashes). The red cross is the point $(\pi/2, 1)$.

The maximum relative error for the six-fold conjugated 7th order series is about 4×10^{-5}.

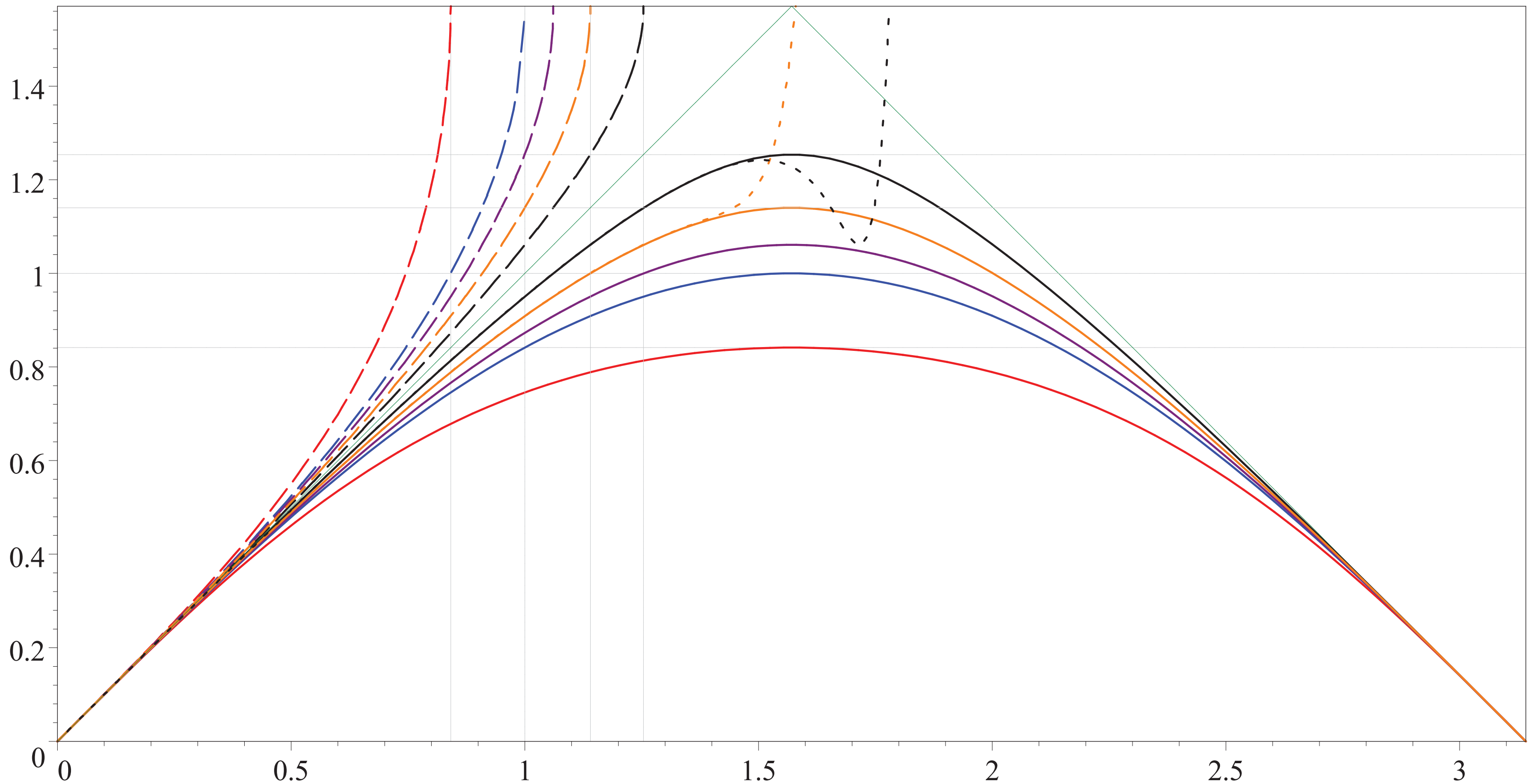
As series, to $O\left(x^{39}\right)$, we find:

$$\operatorname{rin}(x) = x - \frac{1}{12}x^3 - \frac{1}{160}x^5 - \frac{53}{40\,320}x^7 - \frac{23}{71\,680}x^9 - \frac{92\,713}{1277\,337\,600}x^{11} - \frac{742\,031}{79\,705\,866\,240}x^{13} + \frac{594\,673\,187}{167\,382\,319\,104\,000}x^{15} + \frac{329\,366\,540\,401}{91\,055\,981\,592\,576\,000}x^{17} + \frac{104\,491\,760\,828\,591}{62\,282\,291\,409\,321\,984\,000}x^{19} + \frac{1508\,486\,324\,285\,153}{4024\,394\,214\,140\,805\,120\,000}x^{21} - \frac{582\,710\,832\,978\,168\,221}{5882\,770\,031\,248\,492\,462\,080\,000}x^{23} - \frac{1084\,662\,989\,735\,717\,135\,537}{9076\,273\,762\,497\,674\,084\,352\,000\,000}x^{25} - \frac{431\,265\,609\,837\,882\,130\,202\,597}{17\,840\,323\,707\,565\,428\,180\,202\,291\,200\,000}x^{27} + \frac{784\,759\,327\,625\,761\,394\,688\,977\,441}{28\,972\,685\,701\,086\,255\,364\,648\,520\,908\,800\,000}x^{29} + \frac{322\,399\,674\,877\,443\,283\,771\,533\,778\,429}{17\,963\,065\,134\,673\,478\,326\,082\,082\,963\,456\,000\,000}x^{31} - \frac{83\,695\,759\,832\,342\,867\,861\,131\,635\,021\,157}{11\,158\,233\,401\,303\,054\,771\,966\,282\,123\,182\,080\,000\,000}x^{33} - \frac{18\,581\,347\,500\,475\,204\,057\,583\,673\,499\,997\,006\,497}{1354\,386\,370\,250\,164\,788\,221\,267\,324\,111\,840\,870\,400\,000\,000}x^{35} - \frac{31\,078\,612\,581\,904\,734\,618\,917\,797\,435\,079\,542\,343}{52\,291\,091\,164\,441\,144\,866\,977\,625\,383\,100\,638\,822\,400\,000\,000}x^{37} + \frac{4025\,498\,465\,660\,656\,876\,463\,303\,081\,518\,812\,166\,546\,993}{396\,087\,585\,206\,920\,191\,985\,733\,186\,401\,859\,638\,866\,739\,200\,000\,000}x^{39} + O\left(x^{41}\right)$$

$$\operatorname{qin}(x) = x - \frac{1}{24}x^3 - \frac{11}{1920}x^5 - \frac{97}{64\,512}x^7 - \frac{8875}{18\,579\,456}x^9 - \frac{433\,607}{2724\,986\,880}x^{11} - \frac{85\,937\,197}{1700\,391\,813\,120}x^{13} - \frac{19\,340\,942\,627}{1428\,329\,123\,020\,800}x^{15} - \frac{3284\,739\,665\,761}{1554\,022\,085\,846\,630\,400}x^{17} + \frac{5453\,395\,521\,434\,981}{10\,629\,511\,067\,190\,951\,936\,000}x^{19} + \frac{10\,220\,621\,312\,275\,355\,287}{17\,857\,578\,592\,880\,799\,252\,480\,000}x^{21} + \frac{7881\,232\,694\,956\,734\,596\,633}{36\,143\,739\,071\,990\,737\,687\,019\,520\,000}x^{23} + \frac{13\,951\,527\,876\,880\,159\,459\,931}{826\,142\,607\,359\,788\,289\,989\,017\,600\,000}x^{25} - \frac{869\,685\,362\,746\,412\,304\,421\,930\,487}{48\,715\,977\,270\,791\,995\,884\,072\,389\,836\,800\,000}x^{27} + \frac{7373\,357\,881\,673\,321\,431\,225\,463\,783}{1189\,695\,444\,928\,815\,057\,379\,452\,046\,540\,800\,000}x^{29} + \frac{30\,238\,312\,864\,392\,972\,175\,223\,674\,825\,627}{1716\,074\,980\,562\,625\,474\,603\,666\,747\,949\,056\,000\,000}x^{31} + \frac{123\,078\,823\,103\,022\,791\,283\,727\,111\,910\,425\,303}{20\,018\,553\,512\,387\,357\,420\,458\,773\,766\,220\,021\,760\,000\,000}x^{33} - \frac{434\,652\,814\,628\,898\,567\,992\,154\,327\,489\,339\,086\,479\,499}{59\,174\,043\,440\,476\,533\,040\,579\,316\,901\,995\,735\,521\,689\,600\,000\,000}x^{35} - \frac{2219\,316\,435\,479\,862\,370\,647\,640\,243\,739\,603\,493\,250\,242\,937}{315\,279\,303\,450\,858\,968\,040\,206\,600\,453\,833\,278\,859\,562\,188\,800\,000\,000}x^{37} + \frac{546\,987\,077\,306\,820\,361\,992\,453\,535\,155\,367\,973\,773\,862\,088\,481}{207\,663\,967\,872\,965\,773\,615\,816\,080\,832\,258\,186\,342\,164\,961\,689\,600\,000\,000}x^{39} + O\left(x^{41}\right)$$

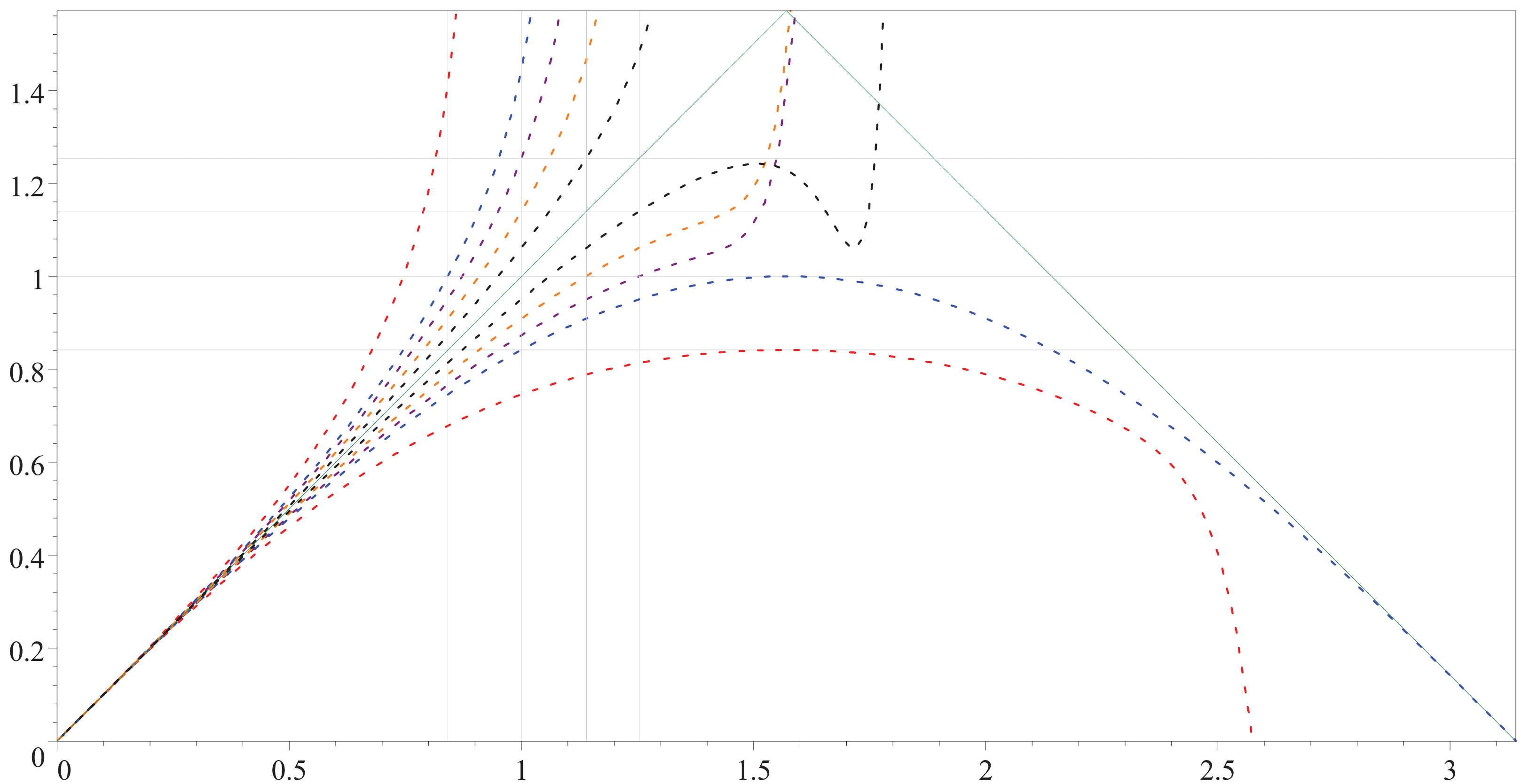
$$\operatorname{arcrin}(x) = x + \frac{1}{12}x^3 + \frac{13}{480}x^5 + \frac{167}{13\,440}x^7 + \frac{38\,617}{5806\,080}x^9 + \frac{989\,077}{255\,467\,520}x^{11} + \frac{944\,469\,221}{398\,529\,331\,200}x^{13} + \frac{5116\,893\,677}{3415\,965\,696\,000}x^{15} + \frac{8009\,684\,322\,739}{8277\,816\,508\,416\,000}x^{17} + \frac{5650\,778\,547\,689\,831}{8897\,470\,201\,331\,712\,000}x^{19} + \frac{22\,093\,952\,731\,139\,969\,213}{52\,317\,124\,783\,830\,466\,560\,000}x^{21} + \frac{80\,433\,921\,544\,323\,338\,879}{283\,127\,969\,418\,376\,642\,560\,000}x^{23} + \frac{4091\,187\,440\,339\,562\,622\,302\,851}{21\,177\,972\,112\,494\,572\,863\,488\,000\,000}x^{25} + \frac{910\,238\,292\,508\,163\,535\,947\,108\,189}{6861\,662\,964\,448\,241\,607\,770\,112\,000\,000}x^{27} + \frac{13\,306\,758\,950\,527\,509\,076\,163\,063\,080\,853}{144\,863\,428\,505\,431\,276\,823\,242\,604\,544\,000\,000}x^{29} + \frac{17\,254\,327\,712\,636\,972\,301\,039\,726\,943\,042\,349}{269\,445\,977\,020\,102\,174\,891\,231\,244\,451\,840\,000\,000}x^{31} + \frac{25\,545\,578\,903\,294\,014\,606\,385\,255\,450\,167\,777\,633}{569\,069\,903\,466\,455\,793\,370\,280\,388\,282\,286\,080\,000\,000}x^{33} + \frac{64\,618\,750\,258\,014\,487\,049\,892\,346\,378\,138\,911\,367}{2042\,815\,038\,084\,713\,104\,406\,134\,727\,167\,180\,800\,000\,000}x^{35} + \frac{26\,950\,499\,741\,891\,615\,526\,387\,651\,295\,686\,986\,010\,188\,239}{1202\,695\,096\,782\,146\,331\,940\,485\,383\,811\,314\,692\,915\,200\,000\,000}x^{37} + \frac{2176\,654\,773\,131\,667\,378\,031\,786\,617\,378\,641\,339\,289\,059}{136\,451\,225\,525\,828\,965\,660\,156\,887\,181\,501\,885\,159\,833\,600\,000}x^{39} + O\left(x^{41}\right)$$

$$\operatorname{arcqin}(x) = x + \frac{1}{24}x^3 + \frac{7}{640}x^5 + \frac{1381}{322\,560}x^7 + \frac{20\,449}{10\,321\,920}x^9 + \frac{40\,742\,441}{40\,874\,803\,200}x^{11} + \frac{1908\,868\,123}{3643\,696\,742\,400}x^{13} + \frac{6029\,839\,810\,061}{21\,424\,936\,845\,312\,000}x^{15} + \frac{3560\,379\,894\,312\,401}{23\,310\,331\,287\,699\,456\,000}x^{17} + \frac{2657\,566\,759\,752\,889\,553}{31\,888\,533\,201\,572\,855\,808\,000}x^{19} + \frac{98\,042\,318\,325\,181\,034\,029}{2142\,909\,431\,145\,695\,910\,297\,600}x^{21} + \frac{917\,344\,555\,532\,911\,475\,292\,007}{36\,143\,739\,071\,990\,737\,687\,019\,520\,000}x^{23} + \frac{413\,160\,788\,801\,033\,971\,847\,312\,689}{28\,914\,991\,257\,592\,590\,149\,615\,616\,000\,000}x^{25} + \frac{5965\,124\,580\,474\,618\,175\,609\,331\,102\,201}{730\,739\,659\,061\,879\,938\,261\,085\,847\,552\,000\,000}x^{27} + \frac{11\,161\,255\,823\,654\,583\,689\,502\,153\,164\,370\,701}{2373\,442\,412\,632\,986\,039\,472\,006\,832\,848\,896\,000\,000}x^{29} + \frac{468\,999\,863\,822\,735\,736\,419\,312\,807\,919\,035\,311}{173\,121\,681\,862\,641\,334\,643\,840\,498\,396\,037\,120\,000\,000}x^{31} + \frac{2004\,389\,689\,545\,536\,737\,500\,771\,201\,629\,581\,091\,413}{1286\,019\,489\,433\,711\,961\,183\,265\,362\,981\,651\,742\,720\,000\,000}x^{33} + \frac{7653\,911\,485\,176\,291\,315\,795\,986\,587\,637\,781\,750\,123\,613}{8453\,434\,777\,210\,933\,291\,511\,330\,985\,999\,390\,788\,812\,800\,000\,000}x^{35} + \frac{39\,112\,520\,438\,585\,288\,910\,926\,125\,480\,581\,329\,807\,170\,101\,577}{72\,756\,762\,334\,813\,608\,009\,278\,446\,258\,576\,910\,506\,052\,812\,800\,000\,000}x^{37} + \frac{1804\,904\,199\,069\,886\,915\,959\,821\,269\,246\,226\,670\,143\,921\,343\,773\,957}{5606\,927\,132\,570\,075\,887\,627\,034\,182\,470\,971\,031\,238\,453\,965\,619\,200\,000\,000}x^{39} + O\left(x^{41}\right)$$



Comparison of $\sin(x) = \sin_{[1]}(x)$ (blue) and $\sin(\sin(x)) = \sin_{[2]}(x)$ (red) to approximations for $\sin_{\left[\frac{3}{4}\right]}(x)$ (purple), $\sin_{\left[\frac{1}{2}\right]}(x)$ (orange), and $\sin_{\left[\frac{1}{4}\right]}(x)$ (black), and their inverses $\arcsin(x) = \sin_{[-1]}(x)$, $\arcsin(\arcsin(x)) = \sin_{[-2]}(x)$, $\sin_{\left[-\frac{3}{4}\right]}(x)$, $\sin_{\left[-\frac{1}{2}\right]}(x)$, and $\sin_{\left[-\frac{1}{4}\right]}(x)$ (dashed), for $0 \le x \le \pi$.

The numerical improvements obtained by acting with inverse roots on $\sin \equiv \sin_{[1]}$ are illustrated in the Figure above. The dotted orange and black curves are obtained directly from the series for $\text{rin} \equiv \sin_{\left[\frac{1}{2}\right]}$ and $\text{qin} \equiv \sin_{\left[\frac{1}{4}\right]}$, respectively, without exploiting the functional relations in (1). The green triangle is actually $\lim_{k \to 0} \sin_{[k]}(x)$.



Same as before, only now *nothing but series* truncated at $O\left(x^{39}\right)$.