Complex Variables

In the calculus of functions of a complex variable there are three fundamental tools, the same fundamental tools as for real variables. Differentiation, Integration, and Power Series. I'll first introduce all three in the context of complex variables, then show the relations between them. The applications of the subject will form the major part of the chapter.

14.1 Differentiation

When you try to differentiate a continuous function is it always differentiable? If it's differentiable once is it differentiable again? The answer to both is no. Take the simple absolute value function of the real variable x.

$$f(x) = |x| = \begin{cases} x & (x \geq 0) \\ -x & (x < 0) \end{cases}$$

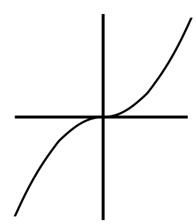
This has a derivative for all x except zero. The limit

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} \longrightarrow \begin{cases} 1 & (x > 0) \\ -1 & (x < 0) \\ ? & (x = 0) \end{cases} \tag{14.1}$$

works for both $x > 0$ and $x < 0$. If $x = 0$ however, you get a different result depending on whether $\Delta x \to 0$ through positive or through negative values.

If you integrate this function,

$$\int_0^x |x'|\,dx' = \begin{cases} x^2/2 & (x \geq 0) \\ -x^2/2 & (x < 0) \end{cases}$$



the result has a derivative everywhere, including the origin, but you can't differentiate it twice. A few more integrations and you can produce a function that you can differentiate 42 times but not 43.

There are functions that are continuous but with no derivative anywhere. They're harder* to construct, but if you grant their existence then you can repeat the preceding manipulation and create a function with any number of derivatives everywhere, but no more than that number anywhere.

For a derivative to exist at a point, the limit Eq. (14.1) must have the same value whether you take the limit from the right or from the left.

Extend the idea of differentiation to complex-valued functions of complex variables. Just change the letter x to the letter $z = x + iy$. Examine a function such as $f(z) = z^2 =$

* Weierstrass surprised the world of mathematics with $\sum_0^\infty a^k \cos(b^k x)$. If $a < 1$ while $ab > 1$ this is continuous but has no derivative anywhere. This statement is much more difficult to prove than it looks.

$x^2 - y^2 + 2ixy$ or $\cos z = \cos x \cosh y + i \sin x \sinh y$. Can you differentiate these (yes) and what does that mean?

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{df}{dz} \tag{14.2}$$

is the appropriate definition, but for it to exist there are even more restrictions than in the real case. For real functions you have to get the same limit as $\Delta x \to 0$ whether you take the limit from the right or from the left. In the complex case there are an infinite number of directions through which Δz can approach zero and you must get the same answer from all directions. This is such a strong restriction that it isn't obvious that *any* function has a derivative. To reassure you that I'm not talking about an empty set, differentiate z^2.

$$\frac{(z + \Delta z)^2 - z^2}{\Delta z} = \frac{2z\Delta z + (\Delta z)^2}{\Delta z} = 2z + \Delta z \longrightarrow 2z$$

It doesn't matter whether $\Delta z = \Delta x$ or $= i\Delta y$ or $= (1 + i)\Delta t$. As long as it goes to zero you get the same answer.

For a contrast take the complex conjugation function, $f(z) = z^* = x - iy$. Try to differentiate that.

$$\frac{(z + \Delta z)^* - z^*}{\Delta z} = \frac{(\Delta z)^*}{\Delta z} = \frac{\Delta r\, e^{-i\theta}}{\Delta r\, e^{i\theta}} = e^{-2i\theta}$$

The polar form of the complex number is more convenient here, and you see that as the distance Δr goes to zero, this difference quotient depends on the direction through which you take the limit. From the right and the left you get $+1$. From above and below ($\theta = \pm\pi/2$) you get -1. The limits aren't the same, so this function has no derivative anywhere. Roughly speaking, the functions that you're familiar with or that are important enough to have names (sin, cos, tanh, Bessel, elliptic, ...) will be differentiable as long as you don't have an explicit complex conjugation in them. Something such as $|z| = \sqrt{z^* z}$ does not have a derivative for any z.

For functions of a real variable, having one or fifty-one derivatives doesn't guarantee you that it has two or fifty-two. The amazing property of functions of a complex variable is that if a function has a single derivative everywhere in the neighborhood of a point then you are guaranteed that it has a infinite number of derivatives. You will also be assured that you can do a power series expansions about that point and that the series will always converge to the function. There are important and useful integration methods that will apply to all these functions, and for a relatively small effort they will open impressively large vistas of mathematics.

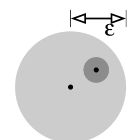
For an example of the insights that you gain using complex variables, consider the function $f(x) = 1/\big(1 + x^2\big)$. This is a perfectly smooth function of x, starting at $f(0) = 1$ and slowing dropping to zero as $x \to \pm\infty$. Look at the power series expansion about $x = 0$ however. This is just a geometric series in $(-x^2)$, so

$$\big(1 + x^2\big)^{-1} = 1 - x^2 + x^4 - x^6 + \cdots$$

This converges only if $-1 < x < +1$. Why such a limitation? The function is infinitely differentiable for all x and is completely smooth throughout its domain. This remains mysterious as long as you think of x as a real number. If you expand your view and consider the function of the complex variable $z = x + iy$, then the mystery disappears. $1/(1 + z^2)$ blows up when

$z \to \pm i$. The reason that the series fails to converge for values of $|x| > 1$ lies in the complex plane, in the fact that at the distance $= 1$ in the i-direction there is a singularity, and in the fact that the domain of convergence is a disk extending out to the nearest singularity.

Definition: A function is said to be analytic at the point z_0 if it is differentiable for every point z in the disk $|z - z_0| < \epsilon$. Here the positive number ϵ may be small, but it is not zero.



Necessarily if f is analytic at z_0 it will also be analytic at every point within the disk $|z - z_0| < \epsilon$. This follows because at any point z_1 within the original disk you have a disk centered at z_1 and of radius $(\epsilon - |z_1 - z_0|)/2$ on which the function is differentiable.

The common formulas for differentiation are exactly the same for complex variables as they are for real variables, and their proofs are exactly the same. For example, the product formula:

$$\begin{aligned}
&\frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z)}{\Delta z} \\
&= \frac{f(z+\Delta z)g(z+\Delta z) - f(z)g(z+\Delta z) + f(z)g(z+\Delta z) - f(z)g(z)}{\Delta z} \\
&= \frac{f(z+\Delta z) - f(z)}{\Delta z} g(z+\Delta z) + f(z)\frac{g(z+\Delta z) - g(z)}{\Delta z}
\end{aligned}$$

As $\Delta z \to 0$, this becomes the familiar $f'g + fg'$. That the numbers are complex made no difference.

For integer powers you can use induction, just as in the real case: $dz/dz = 1$ and

$$\text{If } \frac{dz^n}{dz} = nz^{n-1}, \qquad \text{then use the product rule}$$
$$\frac{dz^{n+1}}{dz} = \frac{d(z^n \cdot z)}{dz} = nz^{n-1} \cdot z + z^n \cdot 1 = (n+1)z^n$$

The other differentiation techniques are in the same spirit. They follow very closely from the definition. For example, how do you handle negative powers? Simply note that $z^n z^{-n} = 1$ and use the product formula. The chain rule, the derivative of the inverse of a function, all the rest, are close to the surface.

14.2 Integration

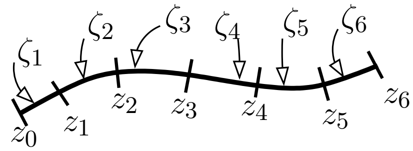
The standard Riemann integral of section 1.6 is

$$\int_a^b f(x)\,dx = \lim_{\Delta x_k \to 0} \sum_{k=1}^{N} f(\xi_k)\Delta x_k$$

The extension of this to complex functions is direct. Instead of partitioning the interval $a < x < b$ into N pieces, you have to specify a curve in the complex plane and partition *it* into N pieces.

The interval is the complex number $\Delta z_k = z_k - z_{k-1}$.

$$\int_C f(z)\,dz = \lim_{\Delta z_k \to 0} \sum_{k=1}^{N} f(\zeta_k)\Delta z_k$$



Just as ξ_k is a point in the k^{th} interval, so is ζ_k a point in the k^{th} interval along the curve C.

How do you evaluate these integrals? Pretty much the same way that you evaluate line integrals in vector calculus. You can write this as

$$\int_C f(z)\,dz = \int \big(u(x,y) + iv(x,y)\big)\big(dx + idy\big) = \int \big[(u\,dx - v\,dy) + i(u\,dy + v\,dx)\big]$$

If you have a parametric representation for the values of $x(t)$ and $y(t)$ along the curve this is

$$\int_{t_1}^{t_2} \big[(u\,\dot{x} - v\,\dot{y}) + i(u\,\dot{y} + v\,\dot{x})\big]dt$$

For example take the function $f(z) = z$ and integrate it around a circle centered at the origin. $x = R\cos\theta$, $y = R\sin\theta$.

$$\begin{aligned}\int z\,dz &= \int \big[(x\,dx - y\,dy) + i(x\,dy + y\,dx)\big] \\ &= \int_0^{2\pi} d\theta R^2 \big[(-\cos\theta\sin\theta - \sin\theta\cos\theta) + i(\cos^2\theta - \sin^2\theta)\big] = 0\end{aligned}$$

Wouldn't it be easier to do this in polar coordinates? $z = re^{i\theta}$.

$$\int z\,dz = \int re^{i\theta}\big[e^{i\theta}dr + ire^{i\theta}d\theta\big] = \int_0^{2\pi} Re^{i\theta} iRe^{i\theta} d\theta = iR^2 \int_0^{2\pi} e^{2i\theta} d\theta = 0 \qquad (14.3)$$

Do the same thing for the function $1/z$. Use polar coordinates.

$$\oint \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{Re^{i\theta}} iRe^{i\theta} d\theta = \int_0^{2\pi} i\,d\theta = 2\pi i \qquad (14.4)$$

This is an important result! Do the same thing for z^n where n is any positive or negative integer, problem 14.1.

Rather than spending time on more examples of integrals, I'll jump to a different subject. The main results about integrals will follow after that (the residue theorem).

14.3 Power (Laurent) Series

The series that concern us here are an extension of the common Taylor or power series, and they are of the form

$$\sum_{-\infty}^{+\infty} a_k (z - z_0)^k \qquad (14.5)$$

The powers can extend through all positive and negative integer values. This is sort of like the Frobenius series that appear in the solution of differential equations, except that here the powers are all integers and they can either have a finite number of negative powers or the powers can go all the way to minus infinity.

The common examples of Taylor series simply represent the case for which no negative powers appear.

$$\sin z = \sum_0^\infty (-1)^k \frac{z^{2k+1}}{(2k+1)!} \qquad \text{or} \qquad J_0(z) = \sum_0^\infty (-1)^k \frac{z^{2k}}{2^{2k}(k!)^2} \qquad \text{or} \qquad \frac{1}{1-z} = \sum_0^\infty z^k$$

If a function has a Laurent series expansion that has a finite number of negative powers, it is said to have a *pole*.

$$\frac{\cos z}{z} = \sum_0^\infty (-1)^k \frac{z^{2k-1}}{(2k)!} \qquad \text{or} \qquad \frac{\sin z}{z^3} = \sum_0^\infty (-1)^k \frac{z^{2k-2}}{(2k+1)!}$$

The *order* of the pole is the size of the largest negative power. These have respectively first order and second order poles.

If the function has an infinite number of negative powers, and the series converges all the way down to (but of course not at) the singularity, it is said to have an *essential singularity*.

$$e^{1/z} = \sum_0^\infty \frac{1}{k!\, z^k} \quad \text{or} \quad \sin\left[t\left(z+\frac{1}{z}\right)\right] = \cdots \quad \text{or} \quad \frac{1}{1-z} = \frac{1}{z}\frac{-1}{1-\frac{1}{z}} = -\sum_1^\infty z^{-k}$$

The first two have essential singularities; the third does not.

It's worth examining some examples of these series and especially in seeing what kinds of singularities they have. In analyzing these I'll use the fact that the familiar power series derived for real variables apply here too. The binomial series, the trigonometric functions, the exponential, many more.

$1/z(z-1)$ has a zero in the denominator for both $z=0$ and $z=1$. What is the full behavior near these two points?

$$\frac{1}{z(z-1)} = \frac{-1}{z(1-z)} = \frac{-1}{z}(1-z)^{-1} = \frac{-1}{z}\big[1+z+z^2+z^3+\cdots\big] = \frac{-1}{z} - 1 - z - z^2 - \cdots$$

$$\begin{aligned}\frac{1}{z(z-1)} &= \frac{1}{(z-1)(1+z-1)} = \frac{1}{z-1}\big[1+(z-1)\big]^{-1} \\ &= \frac{1}{z-1}\big[1+(z-1)+(z-1)^2+(z-1)^3+\cdots\big] = \frac{1}{z-1} + 1 + (z-1) + \cdots\end{aligned}$$

This shows the full Laurent series expansions near these points. Keep your eye on the coefficient of the inverse first power. That term alone plays a crucial role in what will follow.

$\csc^3 z$ near $z=0$:

$$\frac{1}{\sin^3 z} = \frac{1}{\big[z - \frac{z^3}{6} + \frac{z^5}{120} - \cdots\big]^3} = \frac{1}{z^3\big[1 - \frac{z^2}{6} + \frac{z^4}{120} - \cdots\big]^3}$$

$$
\begin{aligned}
&= \frac{1}{z^3}\big[1+x\big]^{-3} = \frac{1}{z^3}\big[1 - 3x + 6x^2 - 10x^3 + \cdots\big] \\
&= \frac{1}{z^3}\left[1 - 3\left(-\frac{z^2}{6} + \frac{z^4}{120} - \cdots\right) + 6\left(-\frac{z^2}{6} + \frac{z^4}{120} - \cdots\right)^2 - \cdots\right] \\
&= \frac{1}{z^3}\left[1 + \frac{z^2}{2} + z^4\left(\frac{1}{6} - \frac{3}{120}\right) + \cdots\right] \\
&= \frac{1}{z^3}\left[1 + \frac{1}{2}z^2 + \frac{17}{120}z^4 + \cdots\right]
\end{aligned}
\tag{14.6}
$$

This has a third order pole, and the coefficient of $1/z$ is $1/2$. Are there any other singularities for this function? Yes, every place that the sine vanishes you have a pole, at $n\pi$. (What is the order of these other poles?) As I commented above, you'll soon see that the coefficient of the $1/z$ term plays a special role, and if that's all that you're looking for you don't have to work this hard. Now that you've seen what various terms do in this expansion, you can stop carrying along so many terms and still get the $1/2z$ term. See problem 14.17

The structure of a Laurent series is such that it will converge in an annulus. Examine the absolute convergence of such a series.

$$
\sum_{-\infty}^{\infty} \big|a_k z^k\big| = \sum_{-\infty}^{-1} \big|a_k z^k\big| + \sum_{0}^{\infty} \big|a_k z^k\big|
$$

The ratio test on the second sum is

$$
\text{if for large enough positive } k, \qquad \frac{|a_{k+1}||z|^{k+1}}{|a_k||z|^k} = \frac{|a_{k+1}|}{|a_k|}|z| \le x < 1 \tag{14.7}
$$

then the series converges. The smallest such x defines the upper bound of the $|z|$ for which the sum of positive powers converges. If $|a_{k+1}|/|a_k|$ has a limit then $|z|_{\text{max}} = \lim |a_k|/|a_{k+1}|$.

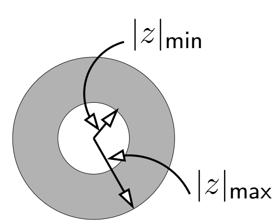
Do the same analysis for the series of negative powers, applying the ratio test.

$$
\text{if for large enough } \textit{negative } \text{k, you have} \qquad \frac{|a_{k-1}||z|^{k-1}}{|a_k||z|^k} = \frac{|a_{k-1}|}{|a_k|}\frac{1}{|z|} \le x < 1 \tag{14.8}
$$

then the series converges. The largest such x defines the *lower* bound of those $|z|$ for which the sum of negative powers converges. If $|a_{k-1}|/|a_k|$ has a limit as $k \to -\infty$ then $|z|_{\text{min}} = \lim |a_{k-1}|/|a_k|$.

If $|z|_{\text{min}} < |z|_{\text{max}}$ then there is a range of z for which the series converges absolutely (and so of course it converges).

$|z|_{\text{min}} < |z| < |z|_{\text{max}}$ an annulus



If either of these series of positive or negative powers is finite, terminating in a polynomial, then respectively $|z|_{\text{max}} = \infty$ or $|z|_{\text{min}} = 0$.

A major result is that when a function is analytic at a point (and so automatically in a neighborhood of that point) then it will have a Taylor series expansion there. The series will converge, and the series will converge to the given function. Is it possible for the Taylor series for a function to converge but not to converge to the expected function? Yes, for functions of a real variable it is. See problem 14.3. The important result is that for analytic functions of a complex variable this cannot happen, and all the manipulations that you would like to do will work. (Well, almost all.)

14.4 Core Properties

There are four closely intertwined facts about analytic functions. Each one implies the other three. For the term "neighborhood" of z_0, take it to mean all points satisfying $|z - z_0| < r$ for some positive r.

1. The function has a single derivative in a neighborhood of z_0.
2. The function has an infinite number of derivatives in a neighborhood of z_0.
3. The function has a power series (positive exponents) expansion about z_0 and the series converges to the specified function in a disk centered at z_0 and extending to the nearest singularity. You can compute the derivative of the function by differentiating the series term-by-term.
4. All contour integrals of the function around closed paths in a neighborhood of z_0 are zero.

Item 3 is a special case of the result about Laurent series. There are no negative powers when the function is analytic at the expansion point.

The second part of the statement, that it's the presence of a singularity that stops the series from converging, requires some computation to prove. The key step in the proof is to show that when the series converges in the neighborhood of a point then you *can* differentiate term-by-term and get the right answer. Since you won't have a derivative at a singularity, the series can't converge there. That important part of the proof is the one that I'll leave to every book on complex variables ever written. *E.g.* Schaum's outline on Complex Variables by Spiegel, mentioned in the bibliography. It's not hard, but it requires attention to detail.

Instead of a direct approach to all these ideas, I'll spend some time showing how they're related to each other. The proofs that these are valid are not all that difficult, but I'm going to spend time on their applications instead.

14.5 Branch Points

The function $f(z) = \sqrt{z}$ has a peculiar behavior. You're so accustomed to it that you may not think of it as peculiar, but only an annoyance that you have to watch out for. It's double valued. The very definition of a function however says that a function is single valued, so what is this? I'll leave the answer to this until later, section 14.7, but for now I'll say that when you encounter this problem you have to be careful of the path along which you move, in order to avoid going all the way around such a point.

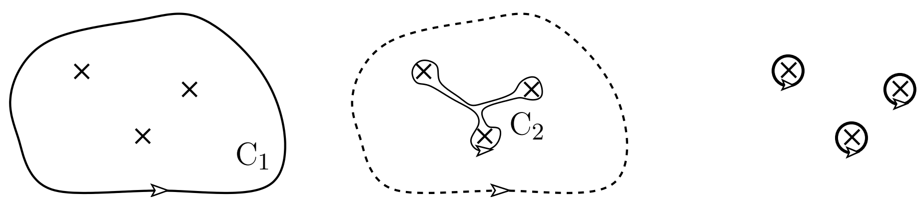
14.6 Cauchy's Residue Theorem

This is *the* fundamental result for applications in physics. If a function has a Laurent series expansion about the point z_0, the coefficient of the term $1/(z-z_0)$ is called the residue of f at z_0. The residue theorem tells you the value of a contour integral around a closed loop in terms of the residues of the function inside the loop.

$$\oint f(z)\,dz = 2\pi i \sum_k \text{Res}(f)\big|_{z_k} \tag{14.9}$$

To make sense of this result I have to specify the hypotheses. The direction of integration is counter-clockwise. Inside and on the simple closed curve defining the path of integration, f is analytic except at isolated points of singularity, where there is a Laurent series expansion. There are no branch points inside the curve. It says that at each singularity z_k inside the contour, find the residue; add them; the result (times $2\pi i$) is the value of the integral on the left. The term "simple" closed curve means that it doesn't cross itself.

Why is this theorem true? The result depends on some of the core properties of analytic functions, especially that fact that you can distort the contour of integration as long as you don't pass over a singularity. If there are several isolated singularities inside a contour (poles or essential singularities), you can contract the contour C_1 to C_2 and then further to loops around the separate singularities. The parts of C_2 other than the loops are pairs of line segments that go in opposite directions so that the integrals along these pairs cancel each other.



The problem is now to evaluate the integrals around the separate singularities of the function, then add them. The function being integrated is analytic in the neighborhoods of the singularities (by assumption they are isolated). That means there is a Laurent series expansion around each, and that it converges all the way down to the singularity itself (though not at it). Now at the singularity z_k you have

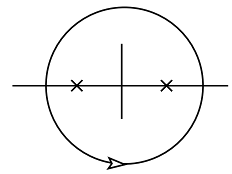
$$\oint \sum_{n=-\infty}^{\infty} a_n (z-z_k)^n$$

The lower limit may be finite, but that just makes it easier. In problem 14.1 you found that the integral of z^n around counterclockwise about the origin is zero unless $n=-1$ in which case it is $2\pi i$. Integrating the individual terms of the series then gives zero from all terms but one, and then it is $2\pi i a_{-1}$, which is $2\pi i$ times the residue of the function at z_k. Add the results from all the singularities and you have the Residue theorem.

Example 1

The integral of $1/z$ around a circle of radius R centered at the origin is $2\pi i$. The Laurent series expansion of this function is trivial — it has only one term. This reproduces Eq. (14.4). It also says that the integral around the same path of $e^{1/z}$ is $2\pi i$. Write out the series expansion of $e^{1/z}$ to determine the coefficient of $1/z$.

Example 2



Another example. The integral of $1/(z^2 - a^2)$ around a circle centered at the origin and of radius $2a$. You can do this integral two ways. First increase the radius of the circle, pushing it out toward infinity. As there are no singularities along the way, the value of the integral is unchanged. The magnitude of the function goes as $1/R^2$ on a large ($R \gg a$) circle, and the circumference is $2\pi R$. the product of these goes to zero as $1/R$, so the value of the original integral (unchanged, remember) is zero.

Another way to do the integral is to use the residue theorem. There are two poles inside the contour, at $\pm a$. Look at the behavior of the integrand near these two points.

$$\frac{1}{z^2 - a^2} = \frac{1}{(z-a)(z+a)} = \frac{1}{(z-a)(2a+z-a)} \approx \text{[near } +a\text{]} \ \frac{1}{2a(z-a)}$$
$$= \frac{1}{(z+a)(z+a-2a)} \approx \text{[near } -a\text{]} \ \frac{1}{-2a(z+a)}$$

The integral is $2\pi i$ times the sum of the two residues.

$$2\pi i \left[\frac{1}{2a} + \frac{1}{-2a}\right] = 0$$

For another example, with a more interesting integral, what is

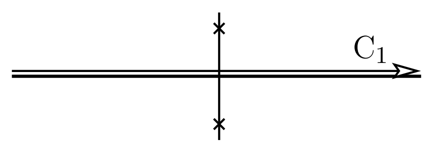
$$\int_{-\infty}^{+\infty} \frac{e^{ikx}\,dx}{a^4 + x^4} \tag{14.10}$$

If these were squares instead of fourth powers, and it didn't have the exponential in it, you could easily find a trigonometric substitution to evaluate it. *This* integral would be formidable though. To illustrate the method, I'll start with that easier example, $\int dx/(a^2 + x^2)$.

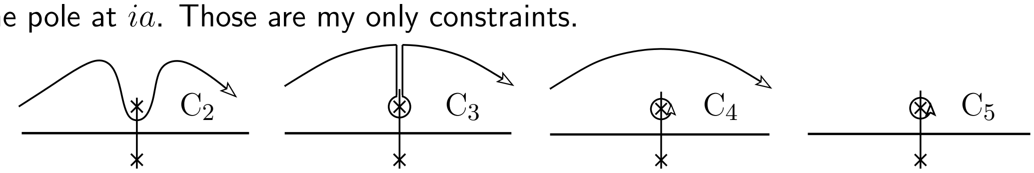
Example 3

The function $1/(a^2 + z^2)$ is singular when the denominator vanishes — when $z = \pm ia$. The integral is the contour integral along the x-axis.

$$\int_{C_1} \frac{dz}{a^2 + z^2} \tag{14.11}$$



The figure shows the two places at which the function has poles, $\pm ia$. The method is to move the contour around and to take advantage of the theorems about contour integrals. First remember that as long as it doesn't move across a singularity, you can distort a contour at will. I will push the contour C_1 up, but I have to leave the endpoints where they are in order not let the contour cross the pole at ia. Those are my only constraints.



As I push the contour from C_1 up to C_2, nothing has changed, and the same applies to C_3. The next two steps however, requires some comment. In C_3 the two straight-line segments that parallel the y-axis are going in opposite directions, and as they are squeezed together, they cancel each other; they are integrals of the same function in reverse directions. In the final step, to C_5, I pushed the contour all the way to $+i\infty$ and eliminated it. How does that happen? On a big circle of radius R, the function $1/(a^2+z^2)$ has a magnitude approximately $1/R^2$. As you push the top curve in C_4 out, forming a big circle, its length is πR. The product of these is π/R, and that approaches zero as $R\to\infty$. All that is left is the single closed loop in C_5, and I evaluate that with the residue theorem.

$$\int_{\mathrm{C}_1} = \int_{\mathrm{C}_5} = 2\pi i \operatorname*{Res}_{z=ia} \frac{1}{a^2+z^2}$$

Compute this residue by examining the behavior near the pole at ia.

$$\frac{1}{a^2+z^2} = \frac{1}{(z-ia)(z+ia)} \approx \frac{1}{(z-ia)(2ia)}$$

Near the point $z=ia$ the value of $z+ia$ is nearly $2ia$, so the coefficient of $1/(z-ia)$ is $1/(2ia)$, and that is the residue. The integral is $2\pi i$ times this residue, so

$$\int_{-\infty}^{\infty} dx\,\frac{1}{a^2+x^2} = 2\pi i\cdot\frac{1}{2ia} = \frac{\pi}{a} \tag{14.12}$$

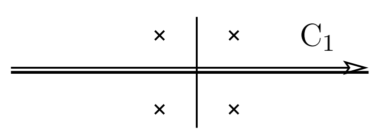
The most obvious check on this result is that it has the correct dimensions. $[dz/z^2] = \mathsf{L}/\mathsf{L}^2 = 1/\mathsf{L}$, a reciprocal length (assuming a is a length). What happens if you push the contour *down* instead of up? See problem 14.10

Example 4

How about the more complicated integral, Eq. (14.10)? There are more poles, so that's where to start. The denominator vanishes where $z^4=-a^4$, or at

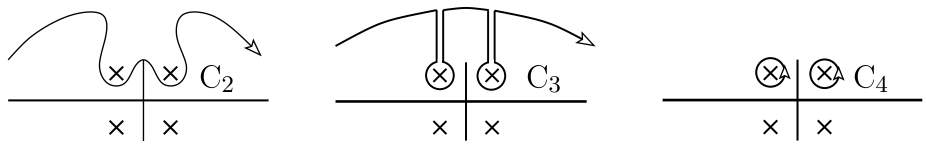
$$z = a\big(e^{i\pi+2in\pi}\big)^{1/4} = ae^{i\pi/4}e^{in\pi/2}$$

$$\int_{\mathrm{C}_1} \frac{e^{ikz}\,dz}{a^4+z^4}$$



I'm going to use the same method as before, pushing the contour past some poles, but I have to be a bit more careful this time. The exponential, not the $1/z^4$, will play the dominant role in the behavior at infinity. If k is positive then if $z=iy$, the exponential $e^{i^2ky}=e^{-ky}\to 0$ as $y\to+\infty$. It will blow up in the $-i\infty$ direction. Of course if k is negative the reverse holds.

Assume $k>0$, then in order to push the contour into a region where I can determine that the integral along it is zero, I have to push it toward $+i\infty$. That's where the exponential drops rapidly to zero. It goes to zero faster than any inverse power of y, so even with the length of the contour going as πR, the combination vanishes.



As before, when you push C_1 up to C_2 and to C_3, nothing has changed, because the contour has crossed no singularities. The transition to C_4 happens because the pairs of straight line segments cancel when they are pushed together and made to coincide. The large contour is pushed to $+i\infty$ where the negative exponential kills it. All that's left is the sum over the two residues at $ae^{i\pi/4}$ and $ae^{3i\pi/4}$.

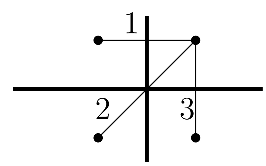
$$\int_{\mathrm{C}_1} = \int_{\mathrm{C}_4} = 2\pi i \sum \mathrm{Res}\, \frac{e^{ikz}}{a^4 + z^4}$$

The denominator factors as

$$a^4 + z^4 = (z - ae^{i\pi/4})(z - ae^{3i\pi/4})(z - ae^{5i\pi/4})(z - ae^{7i\pi/4})$$

The residue at $ae^{i\pi/4} = a(1+i)/\sqrt{2}$ is the coefficient of $1/(z - ae^{i\pi/4})$, so it is

$$\frac{e^{ika(1+i)/\sqrt{2}}}{(ae^{i\pi/4} - ae^{3i\pi/4})(ae^{i\pi/4} - ae^{5i\pi/4})(ae^{i\pi/4} - ae^{7i\pi/4})}$$



Do you have to do a lot of algebra to evaluate this denominator? Maybe you will prefer that to the alternative: *draw a picture.* The distance from the center to a corner of the square is a, so each side has length $a\sqrt{2}$. The first factor in the denominator of the residue is the line labelled "1" in the figure, so it is $a\sqrt{2}$. Similarly the second and third factors (labeled in the diagram) are $2a(1+i)/\sqrt{2}$ and $ia\sqrt{2}$. This residue is then

$$\operatorname*{Res}_{e^{i\pi/4}} = \frac{e^{ika(1+i)/\sqrt{2}}}{(a\sqrt{2})(2a(1+i)/\sqrt{2})(ia\sqrt{2})} = \frac{e^{ika(1+i)/\sqrt{2}}}{a^3 2\sqrt{2}(-1+i)} \tag{14.13}$$

For the other pole, at $e^{3i\pi/4}$, the result is

$$\operatorname*{Res}_{e^{3i\pi/4}} = \frac{e^{ika(-1+i)/\sqrt{2}}}{(-a\sqrt{2})(2a(-1+i)/\sqrt{2})(ia\sqrt{2})} = \frac{e^{ika(-1+i)/\sqrt{2}}}{a^3 2\sqrt{2}(1+i)} \tag{14.14}$$

The final result for the integral Eq. (14.10) is then the sum of these ($\times 2\pi i$)

$$\int_{-\infty}^{+\infty} \frac{e^{ikx}dx}{a^4 + x^4} = 2\pi i\big[(14.13) + (14.14)\big] = \frac{\pi e^{-ka/\sqrt{2}}}{a^3}\cos[(ka/\sqrt{2}) - \pi/4] \tag{14.15}$$

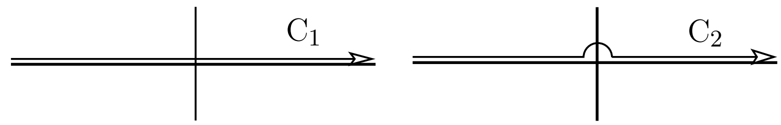
This would be a challenge to do by other means, without using contour integration. It is probably possible, but would be much harder. Does the result make any sense? The dimensions work, because the $[dz/z^4]$ is the same as $1/a^3$. What happens in the original integral if k changes to

$-k$? It's even in k of course. (Really? Why?) This result doesn't look even in k but then it doesn't have to because it applies only for the case that $k > 0$. If you have a negative k you can do the integral again (problem 14.42) and verify that it is even.

Example 5

Another example for which it's not immediately obvious how to use the residue theorem:

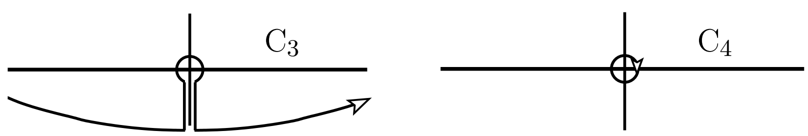
$$\int_{-\infty}^{\infty} dx \frac{\sin ax}{x} \tag{14.16}$$



This function has no singularities. The sine doesn't, and the only place the integrand could have one is at zero. Near that point, the sine itself is linear in x, so $(\sin ax)/x$ is finite at the origin. The trick in using the residue theorem here is to create a singularity where there is none. Write the sine as a combination of exponentials, then the contour integral along C_1 is the same as along C_2, and

$$\int_{\mathrm{C}_1} \frac{e^{iaz} - e^{-iaz}}{2iz} = \int_{\mathrm{C}_2} \frac{e^{iaz} - e^{-iaz}}{2iz} = \int_{\mathrm{C}_2} \frac{e^{iaz}}{2iz} - \int_{\mathrm{C}_2} \frac{e^{-iaz}}{2iz}$$

I had to move the contour away from the origin in anticipation of this splitting of the integrand because I don't want to try integrating *through* this singularity that appears in the last two integrals. In the first form it doesn't matter because there is no singularity at the origin and the contour can move anywhere I want as long as the two points at $\pm\infty$ stay put. In the final two separated integrals it matters very much.



Assume that $a > 0$. In this case, $e^{iaz} \to 0$ as $z \to +i\infty$. For the other exponential, it vanishes toward $-i\infty$. This implies that I can push the contour in the first integral toward $+i\infty$ and the integral over the contour at infinity will vanish. As there are no singularities in the way, that means that the first integral is zero. For the second integral you have to push the contour toward $-i\infty$, and that hangs up on the pole at the origin. That integral is then

$$-\int_{\mathrm{C}_2} \frac{e^{-iaz}}{2iz} = -\int_{\mathrm{C}_4} \frac{e^{-iaz}}{2iz} = -(-2\pi i)\,\mathrm{Res}\,\frac{e^{-iaz}}{2iz} = \pi$$

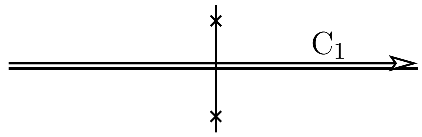
The factor $-2\pi i$ in front of the residue occurs because the integral is over a clockwise contour, thereby changing its sign. Compare the result of problem 5.29(b).

Notice that the result is independent of $a > 0$. (And what if $a < 0$?) You can check this fact by going to the original integral, Eq. (14.16), and making a change of variables. See problem 14.16.

Example 6

What is $\int_0^\infty dx/(a^2 + x^2)^2$? The first observation I'll make is that by dimensional analysis alone, I expect the result to vary as $1/a^3$. Next: the integrand is even, so using the same methods as in the previous examples, extend the integration limits to the whole axis (times $1/2$).

$$\frac{1}{2}\int_{C_1} \frac{dz}{(a^2+z^2)^2}$$



As with Eq. (14.11), push the contour up and it is caught on the pole at $z = ia$. That's curve C_5 following that equation. This time however, the pole is second order, so it take a (little) more work to evaluate the residue.

$$\begin{aligned}\frac{1}{2}\frac{1}{(a^2+z^2)^2} &= \frac{1}{2}\frac{1}{(z-ia)^2(z+ia)^2} = \frac{1}{2}\frac{1}{(z-ia)^2(z-ia+2ia)^2} \\ &= \frac{1}{2}\frac{1}{(z-ia)^2(2ia)^2\big[1+(z-ia)/2ia\big]^2} \\ &= \frac{1}{2}\frac{1}{(z-ia)^2(2ia)^2}\left[1-2\frac{(z-ia)}{2ia}+\cdots\right] \\ &= \frac{1}{2}\frac{1}{(z-ia)^2(2ia)^2} + \frac{1}{2}(-2)\frac{1}{(z-ia)(2ia)^3} + \cdots\end{aligned}$$

The residue is the coefficient of the $1/(z-ia)$ term, so the integral is

$$\int_0^\infty dx/(a^2+x^2)^2 = 2\pi i\cdot(-1)\cdot\frac{1}{(2ia)^3} = \frac{\pi}{4a^3}$$

Is this plausible? The dimensions came out as expected, and to estimate the size of the coefficient, $\pi/4$, look back at the result Eq. (14.12). Set $a = 1$ and compare the π there to the $\pi/4$ here. The range of integration is half as big, so that accounts for a factor of two. The integrands are always less than one, so in the second case, where the denominator is squared, the integrand is always less than that of Eq. (14.12). The integral must be less, and it is. Why less by a factor of two? Dunno, but plot a few points and sketch a graph to see if you believe it. (Or use parametric differentiation to relate the two.)

Example 7

A trigonometric integral: $\int_0^{2\pi} d\theta/(a+b\cos\theta)$. The first observation is that unless $|a|>|b|$ then this denominator will go to zero somewhere in the range of integration (assuming that a and b are real). Next, the result can't depend on the relative sign of a and b, because the change of variables $\theta' = \theta+\pi$ changes the coefficient of b while the periodicity of the cosine means that you can leave the limits alone. I may as well assume that a and b are positive. The trick now is to use Euler's formula and express the cosine in terms of exponentials.

$$\text{Let } z = e^{i\theta}, \qquad \text{then} \qquad \cos\theta = \frac{1}{2}\left[z+\frac{1}{z}\right] \qquad \text{and} \qquad dz = i\,e^{i\theta}d\theta = iz\,d\theta$$

As θ goes from 0 to 2π, the complex variable z goes around the unit circle. The integral is then

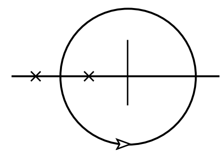
$$\int_0^{2\pi} d\theta\frac{1}{(a+b\cos\theta)} = \int_C \frac{dz}{iz}\,\frac{1}{a+b\left(z+\frac{1}{z}\right)/2}$$

The integrand obviously has some poles, so the first task is to locate them.

$$2az + bz^2 + b = 0 \qquad \text{has roots} \qquad z = \frac{-2a \pm \sqrt{(2a)^2 - 4b^2}}{2b} = z_\pm$$

Because $a > b$, the roots are real. The important question is: Are they inside or outside the unit circle? The roots depend on the ratio $a/b = \lambda$.

$$z_\pm = \left[-\lambda \pm \sqrt{\lambda^2 - 1}\right] \tag{14.17}$$



As λ varies from 1 to ∞, the two roots travel from $-1 \to -\infty$ and from $-1 \to 0$, so z_+ stays inside the unit circle (problem 14.19). The integral is then

$$-\frac{2i}{b}\int_C \frac{dz}{z^2 + 2\lambda z + 1} = -\frac{2i}{b}\int_C \frac{dz}{(z - z_+)(z - z_-)} = -\frac{2i}{b} 2\pi i \operatorname*{Res}_{z=z_+}$$
$$= -\frac{2i}{b} 2\pi i \frac{1}{z_+ - z_-} = \frac{2\pi}{b\sqrt{\lambda^2 - 1}} = \frac{2\pi}{\sqrt{a^2 - b^2}}$$

14.7 Branch Points

Before looking at any more uses of the residue theorem, I have to return to the subject of branch points. They are another type of singularity that an analytic function can have after poles and essential singularities. $\sqrt{z}$ provides the prototype.

The *definition* of the word function, as in section 12.1, requires that it be single-valued. The function $\sqrt{z}$ stubbornly refuses to conform to this. You can get around this in several ways: First, ignore it. Second, change the definition of "function" to allow it to be multiple-valued. Third, change the domain of the function.

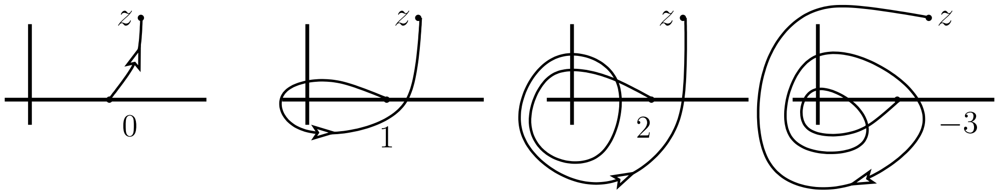
You know I'm not going to ignore it. Changing the definition is not very fruitful. The third method was pioneered by Riemann and is the right way to go.

The complex plane provides a geometric picture of complex numbers, but when you try to handle square roots it becomes a hindrance. It isn't adequate for the task. There are several ways to develop the proper extension, and I'll show a couple of them. The first is a sort of algebraic way, and the second is a geometric interpretation of the first way. There are other, even more general methods, leading into the theory of Riemann Surfaces and their topological structure, but I won't go into those.

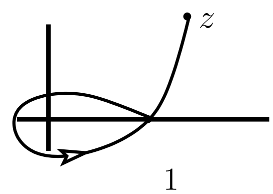
Pick a base point, say $z_0 = 1$, from which to start. This will be a kind of fiduciary point near which I *know* the values of the function. Every other point needs to be referred to this base point. If I state that the square root of z_0 is one, then I haven't run into trouble yet. Take another point $z = re^{i\theta}$ and try to figure out the square root there.

$$\sqrt{z} = \sqrt{re^{i\theta}} = \sqrt{r}\, e^{i\theta/2} \qquad \text{or} \qquad \sqrt{z} = \sqrt{re^{i(\theta+2\pi)}} = \sqrt{r}\, e^{i\theta/2} e^{i\pi}$$

The key question: *How do you get from z_0 to z?* What path did you select?



- In the picture, z appears to be at about $1.5e^{0.6i}$ or so.
- On the path labelled 0, the angle θ starts at zero at z_0 and increases to 0.6 radians, so $\sqrt{r}\,e^{i\theta/2}$ varies continuously from 1 to about $1.25e^{0.3i}$.
- On path labeled 1, angle θ again starts at zero and increases to $0.6+2\pi$, so $\sqrt{r}\,e^{i\theta/2}$ varies continuously from 1 to about $1.25e^{(\pi+0.3)i}$, which is minus the result along path #0.
- On the path labelled 2, angle θ goes from zero to $0.6+4\pi$, and $\sqrt{r}\,e^{i\theta/2}$ varies from 1 to $1.25e^{(2\pi+0.3)i}$ and that is back to the same value as path #0.
- For the path labeled -3, the angle is $0.6-6\pi$, resulting in the same value as path #1.

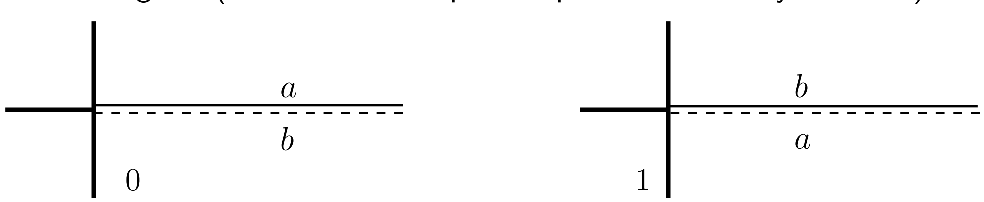


There are two classes of paths from z_0 to z, those that go around the origin an even number of times and those that go around an odd number of times. The "winding number" w is the name given to the number of times that a closed loop goes counterclockwise around a point (positive or negative), and if I take the path #1 and move it slightly so that it passes through z_0, you can more easily see that the only difference between paths 0 and 1 is the single loop around the origin. The value for the square root depends on two variables, z and the winding number of the path. Actually less than this, because it depends only on whether the winding number is even or odd: $\sqrt{z} \to \sqrt{(z,w)}$.

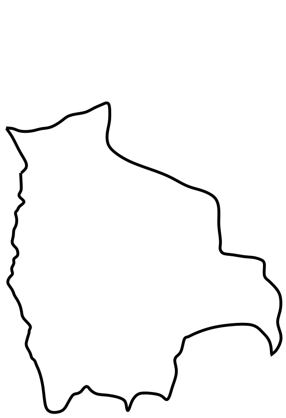
In this notation then $z_0 \to (z_0, 0)$ is the base point, and the square root of that is one. The square root of $(z_0, 1)$ is then minus one. Because the only relevant question about the winding number is whether it is even or odd, it's convenient simply to say that the second argument can take on the values either 0 or 1 and be done with it.

Geometry of Branch Points

How do you picture such a structure? There's a convenient artifice that lets you picture and manipulate functions with branch points. In this square root example, picture two sheets and slice both along some curve starting at the origin and going to infinity. As it's a matter of convenience how you draw the cut I may as well make it a straight line along the x-axis, but any other line (or simple curve) from the origin will do. As these are mathematical planes I'll use mathematical scissors, which have the elegant property that as I cut starting from infinity on the right and proceeding down to the origin, the points that are actually *on* the x-axis are placed on the right side of the cut and the left side of the cut is left open. Indicate this with solid and dashed lines in the figure. (This is not an important point; don't worry about it.)

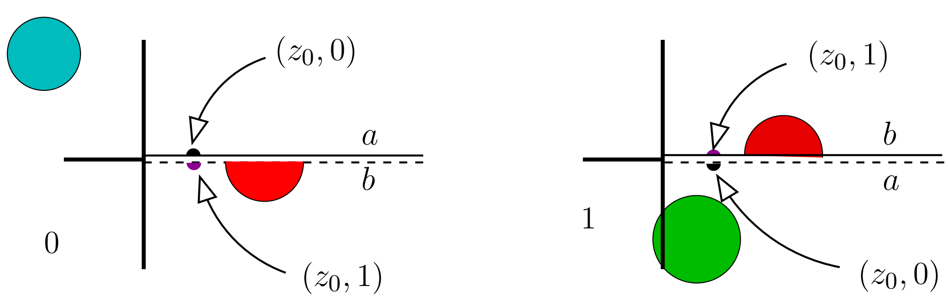


Now sew the sheets together along these cuts. Specifically, sew the top edge from sheet #0 to the bottom edge from sheet #1. I then sew the bottom edge of sheet #0 to the top edge of sheet #1. This sort of structure is called a Riemann surface. How to do this? Do it the same way that you read a map in an atlas of maps. If page 38 of the atlas shows a map with the outline of Brazil and page 27 shows a map with the outline of Bolivia, you can flip back and forth between the two pages and understand that the two maps* represent countries that are touching each other along their common border.



You can see where they fit even though the two countries are not drawn to the same scale. Brazil is a whole lot larger than Bolivia, but where the images fit along the Western border of Brazil and the Eastern border of Bolivia is clear. You are accustomed to doing this with maps, understanding that the right edge of the map on page 27 is the same as the left edge of the map on page 38 — you probably take it for granted. Now you get to do it with Riemann surfaces.

You have two cut planes (two maps), and certain edges are understood to be identified as identical, just as two borders of a geographic map are understood to represent the same line on the surface of the Earth. Unlike the maps above, you will usually draw both to the same scale, but you won't make the cut ragged (no pinking shears) so you need to use some notation to indicate what is attached to what. That's what the letters *a* and *b* are. Side *a* is the same as side *a*. The same for *b*. When you have more complicated surfaces, arising from more complicated functions of the complex variable with many branch points, you will have a fine time sorting out the shape of the surface.

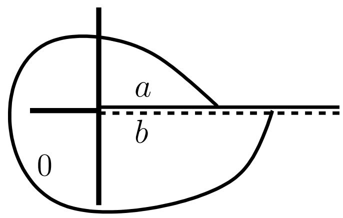


I drew three large disks on this Riemann surface. One is entirely within the first sheet (the first map); a second is entirely within the second sheet. The third disk straddles the two, but is is nonetheless a disk. On a political map this might be disputed territory. Going back to the original square root example, I also indicated the initial point at which to define the value of

* www.worldatlas.com

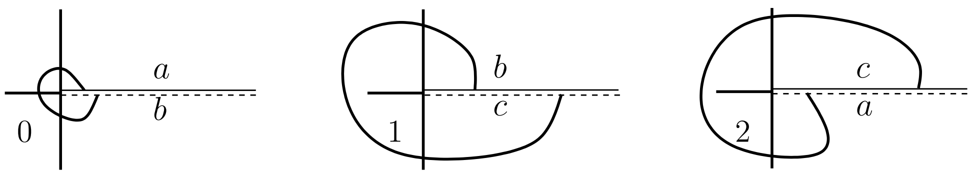
the square root, $(z_0, 0)$, and because a single dot would really be invisible I made it a little disk, which necessarily extends across both sheets.

Here is a picture of a closed loop on this surface. I'll probably not ask you to do contour integrals along such curves though.



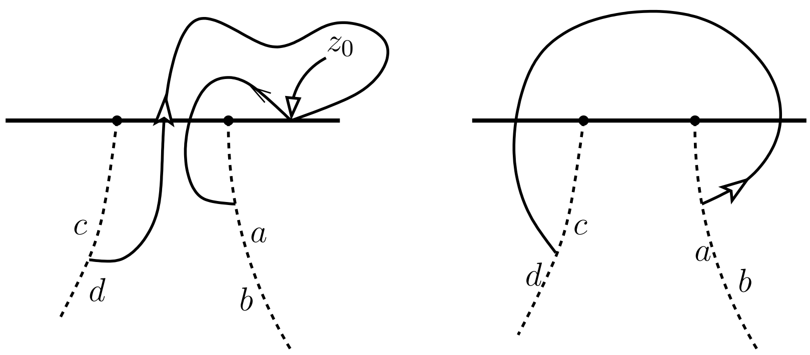
Other Functions

Cube Root Take the next simple step. What about the cube root? Answer: Do exactly the same thing, except that you need three sheets to describe the whole . Again, I'll draw a closed loop. As long as you have only a single branch point it's no more complicated than this.



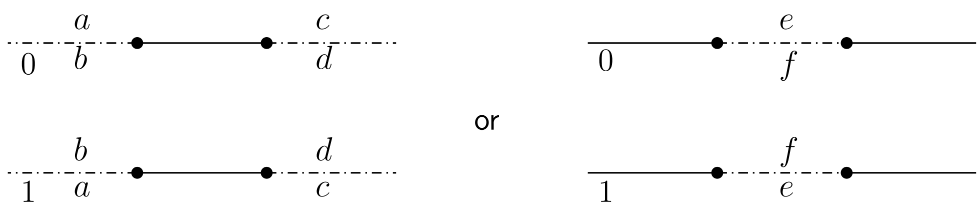
Logarithm How about a logarithm? $\ln z = \ln\left(re^{i\theta}\right) = \ln r + i\theta$. There's a branch point at the origin, but this time, as the angle keeps increasing you never come back to a previous value. This requires an infinite number of sheets. That number isn't any more difficult to handle — it's just like two, only bigger. In this case the whole winding number around the origin comes into play because every loop around the origin, taking you to the next sheet of the surface, adds another $2\pi i w$, and w is any integer from $-\infty$ to $+\infty$. The picture of the surface is like that for the cube root, but with infinitely many sheets instead of three. The complications start to come when you have several branch points.

Two Square Roots Take $\sqrt{z^2 - 1}$ for an example. Many other functions will do just as well. Pick a base point z_0; I'll take 2. (Not two base points, the number 2.) $f(z_0, 0) = \sqrt{3}$. Now follow the function around some loops. This repeats the development as for the single branch, but the number of possible paths will be larger. Draw a closed loop starting at z_0.



Despite the two square roots, you still need only two sheets to map out this surface. I drew the *ab* and *cd* cuts below to keep them out of the way, but they're very flexible. Start the base point and follow the path around the point $+1$; that takes you to the second sheet. You already know that if you go around $+1$ again it takes you back to where you started, so explore a different path: go around -1. Now observe that this function is the *product* of two square roots. Going around the first one introduced a factor of -1 into the function and going around the second branch point will introduce a second identical factor. As $(-1)^2 = +1$, then when you you return to z_0 the function is back at $\sqrt{3}$, you have returned to the base point and this whole loop is closed. If this were the sum of two square roots instead of their product, this wouldn't work. You'll need four sheets to map that surface. See problem 14.22.

These cuts are rather awkward, and now that I know the general shape of the surface it's possible to arrange the maps into a more orderly atlas. Here are two better ways to draw the maps. They're much easier to work with.



I used the dash-dot line to indicate the cuts. In the right pair, the base point is on the right-hand solid line of sheet #0. In the left pair, the base point is on the *c* part of sheet #0. See problem 14.20.

14.8 Other Integrals

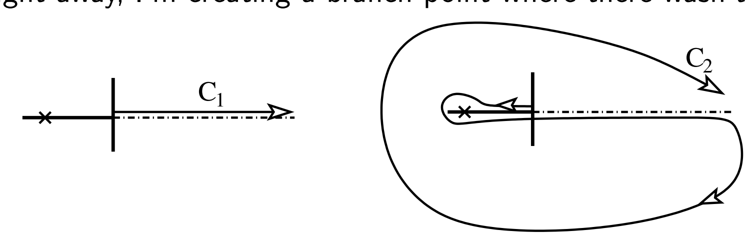
There are many more integrals that you can do using the residue theorem, and some of these involve branch points. In some cases, the integrand you're trying to integrate has a branch point already built into it. In other cases you can pull some tricks and artificially introduce a branch point to facilitate the integration. That doesn't sound likely, but it can happen.

Example 8

The integral $\int_0^\infty dx\, x/(a+x)^3$. You can do this by elementary methods (very easily in fact), but I'll use it to demonstrate a contour method. This integral is from zero to infinity and it isn't even, so the previous tricks don't seem to apply. Instead, consider the integral $(a > 0)$

$$\int_0^\infty dx\, \ln x \,\frac{x}{(a+x)^3}$$

and you see that right away, I'm creating a branch point where there wasn't one before.

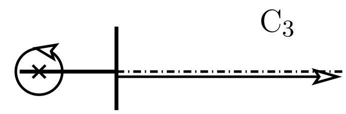


The fact that the logarithm goes to infinity at the origin doesn't matter because it is such a weak singularity that any positive power, even $x^{0.0001}$ times the logarithm, gives a finite limit as $x \to 0$. Take advantage of the branch point that this integrand provides.

$$\int_{C_1} dz\, \ln z \frac{z}{(a+z)^3} = \int_{C_2} dz\, \ln z \frac{z}{(a+z)^3}$$

On C_1 the logarithm is real. After the contour is pushed into position C_2, there are several distinct pieces. A part of C_2 is a large arc that I can take to be a circle of radius R. The size of the integrand is only as big as $(\ln R)/R^2$, and when I multiply this by $2\pi R$, the circumference of the arc, it will go to zero as $R \to \infty$.
The next pieces of C_2 to examine are the two straight lines between the origin and $-a$. The integrals along here are in opposite directions, and there's no branch point intervening, so these two segments simply cancel each other.
What's left is C_3.

$$\int_0^\infty dx\,\ln x\,\frac{x}{(a+x)^3} = \int_{\mathrm{C}_1} = \int_{\mathrm{C}_3}$$
$$= 2\pi i \operatorname*{Res}_{z=-a} + \int_0^\infty dx\,\big(\ln x + 2\pi i\big)\frac{x}{(a+x)^3}$$



Below the positive real axis, that is, below the cut, the logarithm differs from its original value by the constant $2\pi i$. Among all these integrals, the integral with the logarithm on the left side of the equation appears on the right side too. These terms cancel and you're left with

$$0 = 2\pi i \operatorname*{Res}_{z=-a} + \int_0^\infty dx\, 2\pi i \frac{x}{(a+x)^3} \qquad \text{or} \qquad \int_0^\infty dx\,\frac{x}{(a+x)^3} = -\operatorname*{Res}_{z=-a} \ln z \frac{z}{(a+z)^3}$$

This is a third-order pole, so it takes a bit of work. First expand the log around $-a$. Here it's probably easiest to plug into Taylor's formula for the power series and compute the derivatives of $\ln z$ at $-a$.

$$\ln z = \ln(-a) + (z+a)\frac{1}{-a} + \frac{(z+a)^2}{2!}\frac{-1}{(-a)^2} + \cdots$$

Which value of $\ln(-a)$ to take? That answer is dictated by how I arrived at the point $-a$ when I pushed the contour from C_1 to C_2. That is, $\ln a + i\pi$.

$$-\ln z\,\frac{z}{(a+z)^3} = -\left[\ln a + i\pi - \frac{1}{a}(z+a) - \frac{1}{a^2}\frac{(z+a)^2}{2} + \cdots\right]\big[(z+a) - a\big]\frac{1}{(z+a)^3}$$

I'm interested only in the residue, so look only for the coefficient of the power $1/(z+a)$. That is

$$-\left[-\frac{1}{a} - \frac{1}{2a^2}(-a)\right] = \frac{1}{2a}$$

Did you have to do all this work to get this answer? Absolutely not. This falls under the classic heading of using a sledgehammer as a fly swatter. It does show the technique though, and in the process I had an excuse to show that third-order poles needn't be all that intimidating.

14.9 Other Results

Polynomials: There are some other consequences of looking in the complex plane that are very different from any of the preceding. If you did problem 3.11, you realize that the function $e^z = 0$ has no solutions, even in the complex plane. You're used to finding roots of equations such as quadratics and maybe you've encountered the cubic formula too. How do you know that every polynomial even has a root? Maybe there's an order-137 polynomial that has none. No, it doesn't happen. That every polynomial has a root (n of them in fact) is the Fundamental Theorem of Algebra. Gauss proved it, but after the advent of complex variable theory it becomes an elementary exercise.

A polynomial is $f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$. Consider the integral

$$\int_C dz\, \frac{f'(z)}{f(z)}$$

around a large circle. $f'(z) = n a_n z^{n-1} + \cdots$, so this is

$$\int_C dz\, \frac{n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots}{a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0} = \int_C dz\, \frac{n}{z} \frac{1 + \frac{(n-1)a_{n-1}}{n a_n z} + \cdots}{1 + \frac{a_{n-1}}{a_n z} + \cdots}$$

Take the radius of the circle large enough that only the first term in the numerator and the first term in the denominator are important. That makes the integral

$$\int_C dz\, \frac{n}{z} = 2\pi i n$$

It is certainly not zero, so that means that there is a pole inside the loop, and so a root of the denominator.

Function Determined by its Boundary Values: If a function is analytic throughout a simply connected domain and C is a simple closed curve in this domain, then the values of f inside C are determined by the values of f *on* C. Let z be a point inside the contour then I will show

$$\frac{1}{2\pi i} \int_C dz\, \frac{f(z')}{z' - z} = f(z) \tag{14.18}$$

Because f is analytic in this domain I can shrink the contour to be an arbitrarily small curve C_1 around z, and because f is continuous, I can make the curve close enough to z that $f(z') = f(z)$ to any desired accuracy. That implies that the above integral is the same as

$$\frac{1}{2\pi i} f(z) \int_{C_1} dz'\, \frac{1}{z' - z} = f(z)$$

Eq. (14.18) is Cauchy's integral formula, giving the analytic function in terms of its boundary values.

Derivatives: You can differentiate Cauchy's formula any number of times.

$$\frac{d^n f(z)}{dz^n} = \frac{n!}{2\pi i} \int_C dz\, \frac{f(z')}{(z' - z)^{n+1}} \tag{14.19}$$

Entire Functions: An entire function is one that has no singularities anywhere. e^z, polynomials, sines, cosines are such. There's a curious and sometimes useful result about such functions. A bounded entire function is necessarily a constant. For a proof, take two points, z_1 and z_2 and apply Cauchy's integral theorem, Eq. (14.18).

$$f(z_1) - f(z_2) = \frac{1}{2\pi i}\int_C dz\, f(z')\left[\frac{1}{z'-z_1} - \frac{1}{z'-z_2}\right] = \frac{1}{2\pi i}\int_C dz\, f(z')\frac{z_1 - z_2}{(z'-z_1)(z'-z_2)}$$

By assumption, f is bounded, $|f(z)| \le M$. A basic property of complex numbers is that $|u+v| \le |u| + |v|$ for any complex numbers u and v. This means that in the defining sum for an integral,

$$\left|\sum_k f(\zeta_k)\Delta z_k\right| \le \sum_k |f(\zeta_k)||\Delta z_k|, \qquad \text{so} \qquad \left|\int f(z)dz\right| \le \int |f(z)||dz| \tag{14.20}$$

Apply this.

$$|f(z_1) - f(z_2)| \le \int |dz||f(z')|\left|\frac{z_1 - z_2}{(z'-z_1)(z'-z_2)}\right| \le M|z_1 - z_2|\int |dz|\left|\frac{1}{(z'-z_1)(z'-z_2)}\right|$$

On a big enough circle of radius R, this becomes

$$|f(z_1) - f(z_2)| \le M|z_1 - z_2| 2\pi R\frac{1}{R^2} \longrightarrow 0 \quad \text{as } R \to \infty$$

The left side doesn't depend on R, so $f(z_1) = f(z_2)$.

Exercises

1 Describe the shape of the function e^z of the complex variable z. That is, where in the complex plane is this function big? small? oscillating? what is its phase? Make crude sketches to help explain how it behaves as you move toward infinity in many and varied directions. Indicate not only the magnitude, but something about the phase.

2 Same as the preceding but for e^{iz}.

3 Describe the shape of the function z^2. Not just magnitude, other pertinent properties too such as phase, so you know how it behaves.

4 Describe the shape of i/z.

5 Describe the shape of $1/(a^2 + z^2)$. Here you need to show what it does for large distances from the origin and for small. Also near the singularities.

6 Describe the shape of e^{iz^2}.

7 Describe the shape of $\cos z$.

8 Describe the shape of e^{ikz} where k is a real parameter, $-\infty < k < \infty$.

9 Describe the shape of the Bessel function $J_0(z)$. Look up for example Abramowitz and Stegun chapter 9, sections 1 and 6. ($I_0(z) = J_0(iz)$)

Problems

14.1 Explicitly integrate $z^n\,dz$ around the circle of radius R centered at the origin, just as in Eq. (14.4). The number n is any positive, negative, or zero integer.

14.2 Repeat the analysis of Eq. (14.3) but change it to the integral of $z^* dz$.

14.3 For the real-valued function of a real variable,

$$f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0) \\ 0 & (x = 0) \end{cases}$$

Work out all the derivatives at $x = 0$ and so find the Taylor series expansion about zero. Does it converge? Does it converge to f? You did draw a careful graph didn't you? Perhaps even put in some numbers for moderately small x.

14.4 (**a**) The function $1/(z-a)$ has a singularity (pole) at $z = a$. Assume that $|z| < |a|$, and write its series expansion in powers of z/a. Next assume that $|z| > |a|$ and write the series expansion in powers of a/z.
(**b**) In both cases, determine the set of z for which the series is absolutely convergent, replacing each term by its absolute value. Also sketch these sets.
(**c**) Does your series expansion in a/z imply that this function has an essential singularity at $z = 0$? Since you know that it doesn't, what happened?

14.5 The function $1/(1+z^2)$ has a singularity at $z = i$. Write a Laurent series expansion about that point. To do so, note that $1+z^2 = (z-i)(z+i) = (z-i)(2i+z-i)$ and use the binomial expansion to produce the desired series. (Or you can find another, more difficult method.) Use the ratio test to determine the domain of convergence of this series. Specifically, look for (and sketch) the set of z for which the absolute values of the terms form a convergent series.
Ans: $|z-i| < 2$ OR $|z-i| > 2$ depending on which way you did the expansion. If you did one, find the other. If you expanded in powers of $(z-i)$, try expanding in powers of $1/(z-i)$.

14.6 What is $\int_0^i dz/(1-z^2)$? Ans: $i\pi/4$

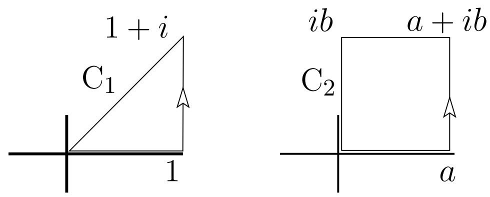
14.7 (**a**) What is a Laurent series expansion about $z = 0$ with $|z| < 1$ to at least four terms for

$$\sin z/z^4 \qquad\qquad e^z/z^2(1-z)$$

(**b**) What is the residue at $z = 0$ for each function?
(**c**) Then assume $|z| > 1$ and find the Laurent series.
Ans: $|z| > 1$: $\sum_{-\infty}^{+\infty} z^n f(n)$, where $f(n) = -e$ if $n < -3$ and $f(n) = -\sum_{n+3}^{\infty} 1/k!$ if $n \geq -3$.

14.8 By explicit integration, evaluate the integrals around the counterclockwise loops:

$$\int_{C_1} z^2\,dz \qquad\qquad \int_{C_2} z^3\,dz$$



14.9 Evaluate the integral along the straight line from a to $a+i\infty$: $\int e^{iz}dz$. Take a to be real.
Ans: ie^{ia}

14.10 (**a**) Repeat the contour integral Eq. (14.11), but this time push the contour *down*, not up.
(**b**) What happens to the same integral if a is negative? And be sure to explain your answer in terms of the contour integrals, even if you see an easier way to do it.

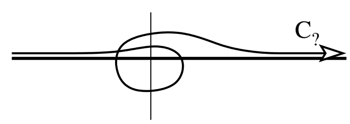
14.11 Carry out all the missing steps starting with Eq. (14.10) and leading to Eq. (14.15).

14.12 Sketch a graph of Eq. (14.15) and for $k < 0$ too. What is the behavior of this function in the neighborhood of $k = 0$? (Careful!)

14.13 In the integration of Eq. (14.16) the contour C_2 had a bump into the upper half-plane. What happens if the bump is into the lower half-plane?

14.14 For the function in problem 14.7, $e^z/z^2(1-z)$, do the Laurent series expansion about $z = 0$, but this time assume $|z| > 1$. What is the coefficient of $1/z$ now? You should have no trouble summing the series that you get for this. Now explain why this result is as it is. Perhaps review problem 14.1.

14.15 In the integration of Eq. (14.16) the contour C_2 had a bump into the upper half-plane, but the original function had no singularity at the origin, so you can instead start with *this* curve and carry out the analysis. What answer do you get?



14.16 Use contour integration to evaluate Eq. (14.16) for the case that $a < 0$.
Next, independently of this, make a change of variables in the original integral Eq. (14.16) in order to see if the answer is independent of a. In this part, consider two cases, $a > 0$ and $a < 0$.

14.17 Recalculate the residue done in Eq. (14.6), but economize your labor. If all that all you really want is the coefficient of $1/z$, keep only the terms that you need in order to get it.

14.18 What is the order of all the other poles of the function $\csc^3 z$, and what is the residue at each pole?

14.19 Verify the location of the roots of Eq. (14.17).

14.20 Verify that the Riemann surfaces work as defined for the function $\sqrt{z^2-1}$ using the alternative maps in section 14.7.

14.21 Map out the Riemann surface for $\sqrt{z(z-1)(z-2)}$. You will need four sheets.

14.22 Map out the Riemann surface for $\sqrt{z} + \sqrt{z-1}$. You will need four sheets.

14.23 Evaluate

$$\int_{\mathrm{C}} dz\, e^{-z} z^{-n}$$

where C is a circle of radius R about the origin.

14.24 Evaluate

$$\int_{\mathrm{C}} dz\,\tan z$$

where C is a circle of radius πn about the origin. Ans: $-4\pi i n$

14.25 Evaluate the residues of these functions at their singularities. a, b, and c are distinct. Six answers: you should be able to do five of them in your head.

$$\text{(a)}\ \frac{1}{(z-a)(z-b)(z-c)} \qquad \text{(b)}\ \frac{1}{(z-a)(z-b)^2} \qquad \text{(c)}\ \frac{1}{(z-a)^3}$$

14.26 Evaluate the residue at the origin for the function

$$\frac{1}{z}e^{z+\frac{1}{z}}$$

The result will be an infinite series, though if you want to express the answer in terms of a standard function you will have to hunt. Ans: $I_0(2) = 2.2796$, a modified Bessel function.

14.27 Evaluate $\int_0^\infty dz/(a^4 + x^4)$, and to check, compare it to the result of Eq. (14.15).

14.28 Show that

$$\int_0^\infty dx\,\frac{\cos bx}{a^2+x^2} = \frac{\pi}{2a}e^{-ab} \qquad (a,\, b > 0)$$

14.29 Evaluate (a real)

$$\int_{-\infty}^{\infty} dx\,\frac{\sin^2 ax}{x^2}$$

Ans: $|a|\pi$

14.30 Evaluate

$$\int_{-\infty}^{\infty} dx\,\frac{\sin^2 bx}{x(a^2+x^2)}$$

14.31 Evaluate the integral $\int_0^\infty dx\,\sqrt{x}\,/(a+x)^2$. Use the ideas of example 8, but without the logarithm. $(a > 0)$ Ans: $\pi/2\sqrt{a}$

14.32 Evaluate

$$\int_0^\infty dx\,\frac{\ln x}{a^2+x^2}$$

(What happens if you consider $(\ln x)^2$?) Ans: $(\pi \ln a)/2a$

14.33 Evaluate $(\lambda > 1)$ by contour integration

$$\int_0^{2\pi} \frac{d\theta}{\big(\lambda + \sin\theta\big)^2}$$

Ans: $2\pi\lambda/(\lambda^2-1)^{3/2}$

14.34 Evaluate

$$\int_0^\pi d\theta\,\sin^{2n}\theta$$

Recall Eq. (2.19). Ans: $\pi\,{}_{2n}C_n\,/2^{2n-1} = \pi(2n-1)!!/(2n)!!$

14.35 Evaluate the integral of problem 14.33 another way. Assume λ is large and expand the integrand in a power series in $1/\lambda$. Use the result of the preceding problem to evaluate the individual terms and then sum the resulting infinite series. Will section 1.2 save you any work? Ans: Still $2\pi\lambda/(\lambda^2-1)^{3/2}$

14.36 Evaluate

$$\int_0^\infty dx\,\cos\alpha x^2 \qquad \text{and} \qquad \int_0^\infty dx\,\sin\alpha x^2 \qquad \text{by considering} \qquad \int_0^\infty dx\,e^{i\alpha x^2}$$

Push the contour of integration toward the 45° line. Ans: $\frac{1}{2}\sqrt{\pi/2\alpha}$

14.37

$$f(z) = \frac{1}{z(z-1)(z-2)} - \frac{1}{z^2(z-1)^2(z-2)^2}$$

What is $\int_C dz\,f(z)$ about the circle $x^2+y^2=9$?

14.38 Derive

$$\int_0^\infty dx \frac{1}{a^3+x^3} = 2\pi\sqrt{3}/9a^2$$

14.39 Go back to problem 3.45 and find the branch points of the inverse sine function.

14.40 What is the Laurent series expansion of $1/(1+z^2)$ for small $|z|$? Again, for large $|z|$? What is the domain of convergence in each case?

14.41 Examine the power series $\sum_0^\infty z^{n!}$. What is its radius of convergence? What is its behavior as you move out from the origin along a radius at a rational angle? That is, $z = re^{i\pi p/q}$ for p and q integers. This result is called a natural boundary.

14.42 Evaluate the integral Eq. (14.10) for the case $k<0$. Combine this with the result in Eq. (14.15) and determine if the overall function is even or odd in k (or neither).

14.43 At the end of section 14.1 several differentiation formulas are mentioned. Derive them.

14.44 Look at the criteria for the annulus of convergence of a Laurent series, Eqs. (14.7) and (14.8), and write down an example of a Laurent series that converges nowhere.

14.45 Verify the integral of example 8 using elementary methods. It will probably take at least three lines to do.

14.46 What is the power series representation for $f(z)=\sqrt{z}$ about the point $1+i$? What is the radius of convergence of this series? In the Riemann surface for this function as described in section 14.7, show the disk of convergence.