

Vector Calculus 2

There's more to the subject of vector calculus than the material in chapter nine. There are a couple of types of line integrals and there are some basic theorems that relate the integrals to the derivatives, sort of like the fundamental theorem of calculus that relates the integral to the anti-derivative in one dimension.

13.1 Integrals

Recall the definition of the Riemann integral from section 1.6.

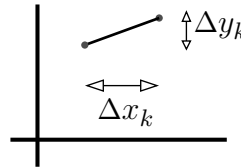
$$\int_a^b dx f(x) = \lim_{\Delta x_k \rightarrow 0} \sum_{k=1}^N f(\xi_k) \Delta x_k \quad (13.1)$$

This refers to a function of a single variable, integrated along that one dimension.

The basic idea is that you divide a complicated thing into little pieces to get an approximate answer. Then you refine the pieces into still smaller ones to improve the answer and finally take the limit as the approximation becomes perfect.

What is the length of a curve in the plane? Divide the curve into a lot of small pieces, then if the pieces are small enough you can use the Pythagorean Theorem to estimate the length of each piece.

$$\Delta \ell_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$



The whole curve then has a length that you estimate to be the sum of all these intervals. Finally take the limit to get the exact answer.

$$\sum_k \Delta \ell_k = \sum \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \longrightarrow \int d\ell = \int \sqrt{dx^2 + dy^2} \quad (13.2)$$

How do you actually *do* this? That will depend on the way that you use to describe the curve itself. Start with the simplest method and assume that you have a parametric representation of the curve:

$$x = f(t) \quad \text{and} \quad y = g(t)$$

Then $dx = \dot{f}(t)dt$ and $dy = \dot{g}(t)dt$, so

$$d\ell = \sqrt{(\dot{f}(t)dt)^2 + (\dot{g}(t)dt)^2} = \sqrt{\dot{f}(t)^2 + \dot{g}(t)^2} dt \quad (13.3)$$

and the integral for the length is

$$\int d\ell = \int_a^b dt \sqrt{\dot{f}(t)^2 + \dot{g}(t)^2}$$

where a and b are the limits on the parameter t . Think of this as $\int d\ell = \int v dt$, where v is the speed.

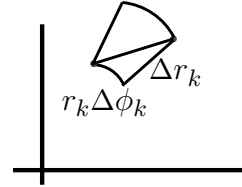
Do the simplest example first. What is the circumference of a circle? Use the parametrization

$$x = R \cos \phi, \quad y = R \sin \phi \quad \text{then} \quad d\ell = \sqrt{(-R \sin \phi)^2 + (R \cos \phi)^2} d\phi = R d\phi \quad (13.4)$$

The circumference is then $\int d\ell = \int_0^{2\pi} R d\phi = 2\pi R$. An ellipse is a bit more of a challenge; see problem 13.3.

If the curve is expressed in polar coordinates you may find another formulation preferable, though in essence it is the same. The Pythagorean Theorem is still applicable, but you have to see what it says in these coordinates.

$$\Delta \ell_k = \sqrt{(\Delta r_k)^2 + (r_k \Delta \phi_k)^2}$$



If this picture doesn't seem to show much of a right triangle, remember there's a limit involved, as Δr_k and $\Delta \phi_k$ approach zero this becomes more of a triangle. The integral for the length of a curve is then

$$\int d\ell = \int \sqrt{dr^2 + r^2 d\phi^2}$$

To actually do this integral you will pick a parameter to represent the curve, and that parameter may even be ϕ itself. For an example, examine one loop of a logarithmic spiral: $r = r_0 e^{k\phi}$.

$$d\ell = \sqrt{dr^2 + r^2 d\phi^2} = \sqrt{(dr/d\phi)^2 + r^2} d\phi$$

The length of the arc from $\phi = 0$ to $\phi = 2\pi$ is

$$\int \sqrt{(r_0 k e^{k\phi})^2 + (r_0 e^{k\phi})^2} d\phi = \int_0^{2\pi} d\phi r_0 e^{k\phi} \sqrt{k^2 + 1} = r_0 \sqrt{k^2 + 1} \frac{1}{k} [e^{2k\pi} - 1]$$

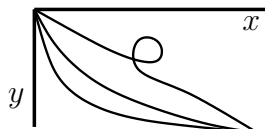
If $k \rightarrow 0$ you can easily check that this give the correct answer. In the other extreme, for large k , you can also check that it is a plausible result, but it's a little harder to see.

Weighted Integrals

The time for a particle to travel along a short segment of a path is $dt = d\ell/v$ where v is the speed. The total time along a path is of course the integral of dt .

$$T = \int dt = \int \frac{d\ell}{v} \quad (13.5)$$

How much time does it take a particle to slide down a curve under the influence of gravity? If the speed is determined by gravity without friction, you can use conservation of energy to compute the speed. I'll use the coordinate y measured downward from the top point of the curve, then



$$mv^2/2 - mgy = E, \quad \text{so} \quad v = \sqrt{(2E/m) + 2gy} \quad (13.6)$$

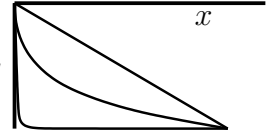
Suppose that this particle starts at rest from $y = 0$, then $E = 0$ and $v = \sqrt{2gy}$. Does the total time to reach a specific point depend on which path you take to get there? Very much so. This will lead to a classic problem called the “brachistochrone.” See section 16.3 for that.

1 Take the straight-line path from $(0, 0)$ to (x_0, y_0) . The path is $x = y \cdot x_0/y_0$.

$$d\ell = \sqrt{dx^2 + dy^2} = dy\sqrt{1 + x_0^2/y_0^2}, \quad \text{so}$$

$$T = \int \frac{d\ell}{v} = \int_0^{y_0} \frac{dy\sqrt{1 + x_0^2/y_0^2}}{\sqrt{2gy}} = \sqrt{1 + x_0^2/y_0^2} \frac{1}{\sqrt{2g}} \frac{1}{2} \sqrt{y_0} = \frac{1}{2} \frac{\sqrt{x_0^2 + y_0^2}}{\sqrt{2gy_0}} \quad (13.7)$$

2 There are an infinite number of possible paths, and another choice of path can give a smaller or a larger time. Take another path for which it's easy to compute the total time. Drop straight down in order to pick up speed, then turn a sharp corner and coast horizontally. Compute the time along this path and it is the sum of two pieces.



$$\int_0^{y_0} \frac{dy}{\sqrt{2gy}} + \int_0^{x_0} \frac{dx}{\sqrt{2gy_0}} = \frac{1}{\sqrt{2g}} \left[\frac{1}{2} \sqrt{y_0} + \frac{x_0}{\sqrt{y_0}} \right] = \frac{1}{\sqrt{2gy_0}} [x_0 + y_0/2] \quad (13.8)$$

Which one takes a shorter time? See problem 13.9.

3 What if the path is a parabola, $x = y^2 \cdot x_0/y_0^2$? It drops rapidly at first, picking up speed, but then takes a more direct route to the end. Use y as the coordinate, then

$$dx = 2y \cdot x_0/y_0^2 dy, \quad \text{and} \quad d\ell = \sqrt{(4y^2 x_0^2/y_0^4) + 1} dy$$

$$T = \int \frac{d\ell}{v} = \int_0^{y_0} \frac{\sqrt{(4y^2 x_0^2/y_0^4) + 1}}{\sqrt{2gy}} dy$$

This is not an integral that you're likely to have encountered yet. I'll refer you to a large table of integrals, where you can perhaps find it under the heading of elliptic integrals.

In more advanced treatments of optics, the time it takes light to travel along a path is of central importance because it is related to the phase of the light wave along that path. In that context however, you usually see it written with an extra factor of the speed of light.

$$cT = \int \frac{c d\ell}{v} = \int n d\ell \quad (13.9)$$

This last form, written in terms of the index of refraction, is called the optical path. Compare problems 2.35 and 2.39.

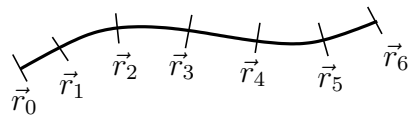
13.2 Line Integrals

Work done on a point mass in one dimension is an integral. If the mass moves in three dimensions and if the force happens to be a constant, then work is simply a dot product:

$$W = \int_{x_i}^{x_f} F_x(x) dx \quad \text{respectively} \quad W = \vec{F} \cdot \Delta \vec{r}$$

The general case for work on a particle moving along a trajectory in space is a line integral. It combines these two equations into a single expression for the work along an arbitrary path for an arbitrary force. There is not then any restriction to the elementary case of constant force.

The basic idea is a combination of Eqs. (13.1) and (13.2). Divide the specified curve into a number of pieces, at the points $\{\vec{r}_k\}$. Between points $k-1$ and k you had the estimate of the arc length as $\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$, but here you need the whole vector from \vec{r}_{k-1} to \vec{r}_k in order to evaluate the work done as the mass moves from one point to the next. Let $\Delta\vec{r}_k = \vec{r}_k - \vec{r}_{k-1}$, then

$$\lim_{|\Delta\vec{r}_k| \rightarrow 0} \sum_{k=1}^N \vec{F}(\vec{r}_k) \cdot \Delta\vec{r}_k = \int \vec{F}(\vec{r}) \cdot d\vec{r}$$

(13.10)

This is the definition of a line integral.

How do you evaluate these integrals? To repeat what happened with Eq. (13.2), that will depend on the way that you use to describe the curve itself. Start with the simplest method and assume that you have a parametric representation of the curve: $\vec{r}(t)$, then $d\vec{r} = \dot{\vec{r}} dt$ and the integral is

$$\int \vec{F}(\vec{r}) \cdot d\vec{r} = \int \vec{F}(\vec{r}(t)) \cdot \dot{\vec{r}} dt$$

This is now an ordinary integral with respect to t . In many specific examples, you may find an easier way to represent the curve, but this is something that you can always fall back on.

In order to see exactly where this is used, start with $\vec{F} = m\vec{a}$. Take the dot product with $d\vec{r}$ and manipulate the expression.

$$\begin{aligned} \vec{F} = m \frac{d\vec{v}}{dt}, \quad \text{so} \quad \vec{F} \cdot d\vec{r} &= m \frac{d\vec{v}}{dt} \cdot d\vec{r} = m \frac{d\vec{v}}{dt} \cdot \frac{d\vec{r}}{dt} dt = m d\vec{v} \cdot \frac{d\vec{r}}{dt} = m \vec{v} \cdot d\vec{v} \\ \text{or} \quad \vec{F} \cdot d\vec{r} &= \frac{m}{2} d(\vec{v} \cdot \vec{v}) \end{aligned} \quad (13.11)$$

The integral of this from an initial point of the motion to a final point is

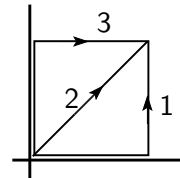
$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \cdot d\vec{r} = \int \frac{m}{2} d(\vec{v} \cdot \vec{v}) = \frac{m}{2} [v_f^2 - v_i^2] \quad (13.12)$$

This is a standard form of the work-energy theorem in mechanics. In most cases you have to specify the whole path, not just the endpoints, so this way of writing the theorem is somewhat misleading. Is it legitimate to manipulate $\vec{v} \cdot d\vec{v}$ as in Eq. (13.11)? Yes. Simply write it in rectangular components as $v_x dv_x + v_y dv_y + v_z dv_z$ and you can integrate each term with no problem; then assemble the result as $v^2/2$.

Example

If $\vec{F} = Axy\hat{x} + B(x^2 + L^2)\hat{y}$, what is the work done going from point $(0, 0)$ to (L, L) along the three different paths indicated?

$$\begin{aligned} \int_{C_1} \vec{F} \cdot d\vec{r} &= \int [F_x \hat{x} + F_y \hat{y}] \cdot [\hat{x} dx + \hat{y} dy] \\ &= \int [F_x dx + F_y dy] = \int_0^L dx 0 + \int_0^L dy B 2L^2 = 2BL^3 \\ \int_{C_2} \vec{F} \cdot d\vec{r} &= \int_0^L dx Ax^2 + \int_0^L dy B(y^2 + L^2) = AL^3/3 + 4BL^3/3 \\ \int_{C_3} \vec{F} \cdot d\vec{r} &= \int_0^L dy B(0 + L^2) + \int_0^L dx AxL = BL^3 + AL^3/2 \end{aligned}$$



Gradient

What is the line integral of a gradient? Recall from section 8.5 and Eq. (8.16) that $df = \text{grad } f \cdot d\vec{r}$. The integral of the gradient is then

$$\int_1^2 \text{grad } f \cdot d\vec{r} = \int df = f_2 - f_1 \quad (13.13)$$

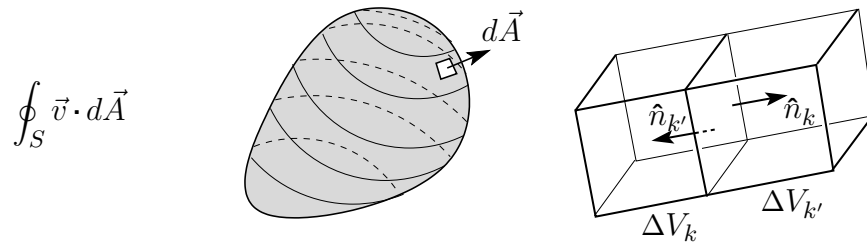
where the indices represent the initial and final points. When you integrate a gradient, you need the function only at its endpoints. The path doesn't matter. Well, almost. See problem 13.19 for a caution.

13.3 Gauss's Theorem

The original definition of the divergence of a vector field is Eq. (9.9),

$$\text{div } \vec{v} = \lim_{V \rightarrow 0} \frac{1}{V} \frac{dV}{dt} = \lim_{V \rightarrow 0} \frac{1}{V} \oint \vec{v} \cdot d\vec{A}$$

Fix a closed surface and evaluate the surface integral of \vec{v} over that surface.



Now divide this volume into a lot of little volumes ΔV_k with individual bounding surfaces S_k . The picture on the right shows just two adjoining pieces of whole volume, but there are many more. If you do the surface integrals of $\vec{v} \cdot d\vec{A}$ over each of these pieces and add all of them, the result is the original surface integral.

$$\sum_k \oint_{S_k} \vec{v} \cdot d\vec{A} = \oint_S \vec{v} \cdot d\vec{A} \quad (13.14)$$

The reason for this is that each interior face of volume V_k is matched with the face of an adjoining volume $V_{k'}$. The latter face will have $d\vec{A}$ pointing in the opposite direction, $\hat{n}_{k'} = -\hat{n}_k$, so when you add all the interior surface integrals they cancel. All that's left is the surface on the outside and the sum over all *those* faces is the original surface integral.

In the equation (13.14) multiply and divide every term in the sum by the volume ΔV_k .

$$\sum_k \left[\frac{1}{\Delta V_k} \oint_{S_k} \vec{v} \cdot d\vec{A} \right] \Delta V_k = \oint_S \vec{v} \cdot d\vec{A}$$

Now increase the number of subdivisions, finally taking the limit as all the ΔV_k approach zero. The quantity inside the brackets becomes the definition of the divergence of \vec{v} and you then get

$$\text{Gauss's Theorem:} \quad \int_V \text{div } \vec{v} dV = \oint_S \vec{v} \cdot d\vec{A} \quad (13.15)$$

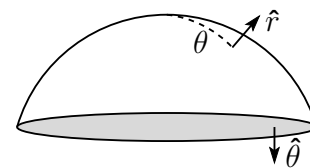
This* is Gauss's theorem, the divergence theorem.

* You will sometimes see the notation ∂V instead of S for the boundary surface surrounding the volume V . Also ∂A instead of C for the boundary curve surrounding the area A . It is probably a better and more consistent notation, but it isn't yet as common in physics books.

Example

Verify Gauss's Theorem for the solid hemisphere, $r \leq R$, $0 \leq \theta \leq \pi/2$, $0 \leq \phi \leq 2\pi$. Use the vector field

$$\vec{F} = \hat{r} \alpha r^2 \sin \theta + \hat{\theta} \beta r \theta^2 \cos^2 \phi + \hat{\phi} \gamma r \sin \theta \cos^2 \phi \quad (13.16)$$



Doing the surface integral on the hemisphere $\hat{n} = \hat{r}$, and on the bottom flat disk you have $\hat{n} = \hat{\theta}$. The surface integral is then assembled from two pieces,

$$\begin{aligned} \oint \vec{F} \cdot d\vec{A} &= \int_{\text{hemisph}} \hat{r} \alpha r^2 \sin \theta \cdot \hat{r} dA + \int_{\text{disk}} \hat{\theta} \beta r \theta^2 \cos^2 \phi \cdot \hat{\theta} dA \\ &= \int_0^{\pi/2} R^2 \sin \theta d\theta \int_0^{2\pi} d\phi \alpha R^2 \sin \theta + \int_0^R r dr \int_0^{2\pi} d\phi \beta r (\pi/2)^2 \cos^2 \phi \\ &= \alpha \pi^2 R^4 / 2 + \beta \pi^3 R^3 / 12 \end{aligned} \quad (13.17)$$

Now do the volume integral of the divergence, using Eq. (9.16).

$$\begin{aligned} \text{div } \vec{F} &= \frac{1}{r^2} \frac{\partial}{\partial r} \alpha r^4 \sin \theta + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \beta r \theta^2 \sin \theta \cos^2 \phi + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \gamma r \sin \theta \cos^2 \phi \\ &= 4\alpha r \sin \theta + \beta \cos^2 \phi [2\theta + \theta^2 \cot \theta] + 2\gamma \sin \phi \cos \phi \end{aligned}$$

The γ term in the volume integral is zero because the $2 \sin \phi \cos \phi = \sin 2\phi$ factor averages to zero over all ϕ .

$$\begin{aligned} \int_0^R dr r^2 \int_0^{\pi/2} \sin \theta d\theta \int_0^{2\pi} d\phi [4\alpha r \sin \theta + \beta \cos^2 \phi [2\theta + \theta^2 \cot \theta]] \\ = 4\alpha \cdot \frac{R^4}{4} \cdot 2\pi \cdot \frac{\pi}{4} + \beta \cdot \frac{R^3}{3} \cdot \pi \cdot \int_0^{\pi/2} d\theta \sin \theta [2\theta + \theta^2 \cot \theta] \\ = \alpha \pi^2 R^4 / 2 + \beta \pi^3 R^3 / 12 \end{aligned}$$

The last integration used parametric differentiation starting from $\int_0^{\pi/2} d\theta \cos k\theta$, with differentiation with respect to k .

13.4 Stokes' Theorem

The expression for the curl in terms of integrals is Eq. (9.17),

$$\text{curl } \vec{v} = \lim_{V \rightarrow 0} \frac{1}{V} \oint d\vec{A} \times \vec{v} \quad (13.18)$$

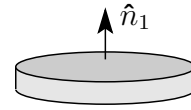
Use the same reasoning that leads from the definition of the divergence to Eqs. (13.14) and (13.15) (see problem 13.6), and this leads to the analog of Gauss's theorem, but with cross products.

$$\oint_S d\vec{A} \times \vec{v} = \int_V \text{curl } \vec{v} dV \quad (13.19)$$

This isn't yet in a form that is all that convenient, and a special case is both easier to interpret and more useful in applications. First apply it to a particular volume, one that is very thin and small. Take a tiny disk of height Δh , with top and bottom area ΔA_1 . Let \hat{n}_1 be the unit normal vector out of the

top area. For small enough values of these dimensions, the right side of Eq. (13.18) is simply the value of the vector $\text{curl } \vec{v}$ inside the volume $\Delta A_1 \Delta h$ itself.

$$\oint_S d\vec{A} \times \vec{v} = \int_V \text{curl } \vec{v} dV = \text{curl } \vec{v} \Delta A_1 \Delta h$$

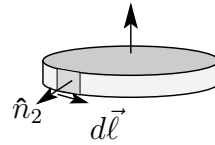


Take the dot product of both sides with \hat{n}_1 , and the parts of the surface integral from the top and the bottom faces disappear. That's just the statement that on the top and the bottom, $d\vec{A}$ is in the direction of $\pm \hat{n}_1$, so the cross product makes $d\vec{A} \times \vec{v}$ perpendicular to \hat{n}_1 .

I'm using the subscript ₁ for the top surface and I'll use ₂ for the surface around the edge. Otherwise it's too easy to get the notation mixed up.

Now look at $d\vec{A}_2 \times \vec{v}$ around the thin edge. The element of area has height Δh and length $\Delta \ell$ along the arc. Call \hat{n}_2 the unit normal out of the edge.

$$\Delta \vec{A}_2 = \Delta h \Delta \ell \hat{n}_2$$



The product $\hat{n}_1 \cdot \Delta \vec{A}_2 \times \vec{v} = \hat{n}_1 \cdot \hat{n}_2 \times \vec{v} \Delta h \Delta \ell = \hat{n}_1 \times \hat{n}_2 \cdot \vec{v} \Delta h \Delta \ell$, using the property of the triple scalar product. The product $\hat{n}_1 \times \hat{n}_2$ is in the direction along the arc of the edge, so

$$\hat{n}_1 \times \hat{n}_2 \Delta \ell = \Delta \vec{\ell} \quad (13.20)$$

Put all these pieces together and you have

$$\hat{n}_1 \cdot \oint_S d\vec{A} \times \vec{v} = \oint_C \vec{v} \cdot d\vec{\ell} \Delta h = \hat{n}_1 \cdot \text{curl } \vec{v} \Delta A_1 \Delta h$$

Divide by $\Delta A_1 \Delta h$ and take the limit as $\Delta A_1 \rightarrow 0$. Recall that all the manipulations above work only under the assumption that you take this limit.

$$\hat{n}_1 \cdot \text{curl } \vec{v} = \lim_{\Delta A \rightarrow 0} \frac{1}{\Delta A} \oint_C \vec{v} \cdot d\vec{\ell} \quad (13.21)$$

You will sometimes see this equation (13.21) taken as the definition of the curl, and it does have an intuitive appeal. The one drawback to doing this is that it isn't at all obvious that the thing on the right-hand side is the dot product of \hat{n}_1 with anything. It is, because I deduced that fact from the vectors in Eq. (13.19), but if you use Eq. (13.21) as your starting point you have some proving to do.

This form is easier to interpret than was the starting point with a volume integral. The line integral of $\vec{v} \cdot d\vec{\ell}$ is called the circulation of \vec{v} around the loop. Divide this by the area of the loop and take the limit as the area goes to zero and you then have the "circulation density" of the vector field. The component of the curl along some direction is then the circulation density around that direction. Notice that the equation (13.20) dictates the right-hand rule that the direction of integration around the loop is related to the direction of the normal \hat{n}_1 .

Stokes' theorem follows in a few lines from Eq. (13.21). Pick a surface A with a boundary C (or ∂A in the other notation). The surface doesn't have to be flat, but you have to be able to tell one

side from the other.* From here on I'll imitate the procedure of Eq. (13.14). Divide the surface into a lot of little pieces ΔA_k , and do the line integral of $\vec{v} \cdot d\vec{\ell}$ around each piece. Add all these pieces and the result is the whole line integral around the outside curve.

$$\sum_k \oint_{C_k} \vec{v} \cdot d\vec{\ell} = \oint_C \vec{v} \cdot d\vec{\ell} \quad \text{[Diagram of a sphere with a grid and two adjacent regions labeled } k \text{ and } k' \text{ with arrows indicating opposite directions]} \quad (13.22)$$

As before, on each interior boundary between area ΔA_k and the adjoining $\Delta A_{k'}$, the parts of the line integrals on the common boundary cancel because the directions of integration are opposite to each other. All that's left is the curve on the outside of the whole loop, and the sum over *those* intervals is the original line integral.

Multiply and divide each term in the sum (13.22) by ΔA_k and you have

$$\sum_k \left[\frac{1}{\Delta A_k} \oint_{C_k} \vec{v} \cdot d\vec{\ell} \right] \Delta A_k = \oint_C \vec{v} \cdot d\vec{\ell} \quad (13.23)$$

Now increase the number of subdivisions of the surface, finally taking the limit as all the $\Delta A_k \rightarrow 0$, and the quantity inside the brackets becomes the normal component of the curl of \vec{v} by Eq. (13.21). The limit of the sum is the definition of an integral, so

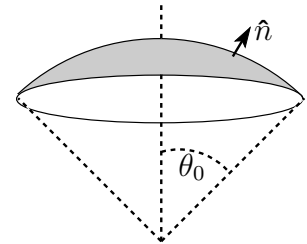
$$\text{Stokes' Theorem:} \quad \int_A \text{curl } \vec{v} \cdot d\vec{A} = \oint_C \vec{v} \cdot d\vec{\ell} \quad (13.24)$$

What happens if the vector field \vec{v} is the gradient of a function, $\vec{v} = \nabla f$? By Eq. (13.13) the line integral in (13.24) depends on just the endpoints of the path, but in this integral the initial and final points are the same. That makes the integral zero: $f_1 - f_1$. That implies that the surface integral on the left is zero no matter what the surface spanning the contour is, and that can happen only if the thing being integrated is itself zero. $\text{curl grad } f = 0$. That's one of the common vector identities in problem 9.36. Of course this statement requires the usual assumption that there are no singularities of \vec{v} within the area.

Example

Verify Stokes' theorem for that part of a spherical surface $r = R$, $0 \leq \theta \leq \theta_0$, $0 \leq \phi < 2\pi$. Use for this example the vector field

$$\vec{F} = \hat{r} Ar^2 \sin \theta + \hat{\theta} Br\theta^2 \cos \phi + \hat{\phi} Cr \sin \theta \cos^2 \phi \quad (13.25)$$



To compute the curl of \vec{F} , use Eq. (9.33), getting

$$\begin{aligned} \nabla \times \vec{F} &= \hat{r} \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (\sin \theta Cr \sin \theta \cos^2 \phi) - \frac{\partial}{\partial \phi} (Br\theta^2 \cos \phi) \right) + \dots \\ &= \hat{r} \frac{1}{r \sin \theta} (Cr \cos^2 \phi 2 \sin \theta \cos \theta + Br\theta^2 \sin \phi) + \dots \end{aligned} \quad (13.26)$$

* That means no Klein bottles or Möbius strips.

I need only the \hat{r} component of the curl because the surface integral uses only the normal (\hat{r}) component. The surface integral of this has the area element $dA = r^2 \sin \theta \, d\theta \, d\phi$.

$$\begin{aligned} \int \text{curl } \vec{F} \cdot d\vec{A} &= \int_0^{\theta_0} R^2 \sin \theta \, d\theta \int_0^{2\pi} d\phi \frac{1}{R \sin \theta} (CR \cos^2 \phi \, 2 \sin \theta \cos \theta + BR\theta^2 \sin \phi) \\ &= R^2 \int_0^{\theta_0} d\theta \int_0^{2\pi} d\phi \, 2C \cos^2 \phi \sin \theta \cos \theta \\ &= R^2 2C\pi \sin^2 \theta_0 / 2 = CR^2\pi \sin^2 \theta_0 \end{aligned}$$

The other side of Stokes' theorem is the line integral around the circle at angle θ_0 .

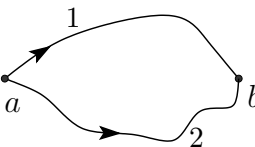
$$\begin{aligned} \oint \vec{F} \cdot d\vec{\ell} &= \int_0^{2\pi} r \sin \theta_0 \, d\phi \, Cr \sin \theta \cos^2 \phi \\ &= \int_0^{2\pi} d\phi \, CR^2 \sin^2 \theta_0 \cos^2 \phi \\ &= CR^2 \sin^2 \theta_0 \pi \end{aligned} \tag{13.27}$$

and the two sides of the theorem agree. Check! Did I get the overall signs right? The direction of integration around the loop matters. A further check: If $\theta_0 = \pi$, the length of the loop is zero and both integrals give zero as they should.

Conservative Fields

An immediate corollary of Stokes' theorem is that if the curl of a vector field is zero throughout a region then line integrals are independent of path in that region. To state it a bit more precisely, in a volume for which any closed path can be shrunk to a point without leaving the region, if the curl of \vec{v} equals zero, then $\int_a^b \vec{F} \cdot d\vec{r}$ depends on the endpoints of the path, and not on how you get there.

To see why this follows, take two integrals from point a to point b .

$$\int_1 \vec{v} \cdot d\vec{r} \quad \text{and} \quad \int_2 \vec{v} \cdot d\vec{r}$$


The difference of these two integrals is

$$\int_1 \vec{v} \cdot d\vec{r} - \int_2 \vec{v} \cdot d\vec{r} = \oint \vec{v} \cdot d\vec{r}$$

This equation happens because the minus sign is the same thing that you get by integrating in the reverse direction. For a field with $\nabla \times \vec{v} = 0$, Stokes' theorem says that this closed path integral is zero, and the statement is proved.

What was that fussy-sounding statement "for which any closed path can be shrunk to a point without leaving the region?" Consider the vector field in three dimensions, written in rectangular and cylindrical coordinates,

$$\vec{v} = A(x\hat{y} - y\hat{x})/(x^2 + y^2) = A\hat{\phi}/r \tag{13.28}$$

You can verify (in either coordinate system) that its curl is zero — except for the z -axis, where it is singular. A closed loop line integral that doesn't encircle the z -axis will be zero, but if it does go around the axis then it is not. The vector's direction $\hat{\theta}$ always points counterclockwise around the axis. See

problem 13.17. If you have a loop that encloses the singular line, then you can't shrink the loop without its getting hung up on the axis.

The converse of this theorem is also true. If every closed-path line integral of \vec{v} is zero, and if the derivatives of \vec{v} are continuous, then its curl is zero. Stokes' theorem tells you that every surface integral of $\nabla \times \vec{v}$ is zero, so you can pick a point and a small $\Delta\vec{A}$ at this point. For small enough area whatever the curl is, it won't change much. The integral over this small area is then $\nabla \times \vec{v} \cdot \Delta\vec{A}$, and by assumption this is zero. It's zero for all values of the area vector. The only vector whose dot product with all vectors is zero is itself the zero vector.

Potentials

The relation between the vanishing curl and the fact that the line integral is independent of path leads to the existence of potential functions.

If $\text{curl } \vec{F} = 0$ in a simply-connected domain (that's one for which any closed loop can be shrunk to a point), then I can write \vec{F} as a gradient, $-\text{grad } f$. The minus sign is conventional. I've already constructed the answer (almost), and to complete the calculation note that line integrals are independent of path in such a domain, and that means that the integral

$$\int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r} \quad (13.29)$$

is a function of the two endpoints alone. Fix \vec{r}_0 and treat this as a function of the upper limit \vec{r} . Call it $-f(\vec{r})$. The defining equation for the gradient is Eq. (8.16),

$$df = \text{grad } f \cdot d\vec{r}$$

How does the integral (13.29) change when you change \vec{r} a bit?

$$\int_{\vec{r}_0}^{\vec{r}+d\vec{r}} \vec{F} \cdot d\vec{r} - \int_{\vec{r}_0}^{\vec{r}} \vec{F} \cdot d\vec{r} = \int_{\vec{r}}^{\vec{r}+d\vec{r}} \vec{F} \cdot d\vec{r} = F \cdot d\vec{r}$$

This is $-df$ because I called this integral $-f(\vec{r})$. Compare the last two equations and because $d\vec{r}$ is arbitrary you immediately get

$$\vec{F} = -\text{grad } f \quad (13.30)$$

I used this equation in section 9.9, stating that the existence of the gravitational potential energy followed from the fact that $\nabla \times \vec{g} = 0$.

Vector Potentials

This is not strictly under the subject of conservative fields, but it's a convenient place to discuss it anyway. When a vector field has zero curl then it's a gradient. When a vector field has zero divergence then it's a curl. In both cases the converse is simple, and it's what you see first: $\nabla \times \nabla f = 0$ and $\nabla \cdot \nabla \times \vec{A} = 0$ (problem 9.36). In Eqs. (13.29) and (13.30) I was able to construct the function f because $\nabla \times \vec{F} = 0$. It is also possible, if $\nabla \cdot \vec{F} = 0$, to construct the function \vec{A} such that $\vec{F} = \nabla \times \vec{A}$.

In both cases, there are extra conditions needed for the statements to be completely true. To conclude that a conservative field ($\nabla \times \vec{F} = 0$) is a gradient requires that the domain be simply-connected, allowing the line integral to be completely independent of path. To conclude that a field satisfying $\nabla \cdot \vec{F} = 0$ can be written as $\vec{F} = \nabla \times \vec{A}$ requires something similar: that all closed *surfaces* can be shrunk to a point. This statement is not so easy to prove, and the explicit construction of \vec{A} from \vec{F} is not very enlightening.

You can easily verify that $\vec{A} = \vec{B} \times \vec{r}/2$ is a vector potential for the uniform field \vec{B} . Neither the scalar potential nor the vector potential are unique. You can always add a constant to a scalar potential because the gradient of a scalar is zero and it doesn't change the result. For the vector potential you can add the gradient of an arbitrary function because that doesn't change the curl.

$$\vec{F} = -\nabla(f + C) = -\nabla f, \quad \text{and} \quad \vec{B} = \nabla \times (\vec{A} + \nabla f) = \nabla \times \vec{A} \quad (13.31)$$

13.5 Reynolds Transport Theorem

When an integral has limits that are functions of time, how do you differentiate it? That's pretty easy for one-dimensional integrals, as in Eqs. (1.19) and (1.21).

$$\frac{d}{dt} \int_{f_1(t)}^{f_2(t)} dx g(x, t) = \int_{f_1(t)}^{f_2(t)} dx \frac{\partial g(x, t)}{\partial t} + g(f_2(t), t) \frac{df_2(t)}{dt} - g(f_1(t), t) \frac{df_1(t)}{dt} \quad (13.32)$$

One of Maxwell's equations for electromagnetism is

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (13.33)$$

Integrate this equation over the surface S .

$$\int_S \nabla \times \vec{E} \cdot d\vec{A} = \int_C \vec{E} \cdot d\vec{\ell} = \int_S -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} \quad (13.34)$$

This used Stokes' theorem, and I would like to be able to pull the time derivative out of the integral, but can I? If the surface is itself time independent then the answer is yes, but what if it isn't? What if the surface integral has a surface that is moving? Can this happen? That's how generators work, and you wouldn't be reading this now without the power they provide. The copper wire loops are rotating at high speed, and it is this motion that provides the EMF.

I'll work backwards and compute the time derivative of a surface integral, allowing the surface itself to move. To do this, I'll return to the definition of a derivative. The time variable appears in two places, so use the standard trick of adding and subtracting a term, just as in section 1.5. Call Φ the flux integral, $\int \vec{B} \cdot d\vec{A}$.

$$\begin{aligned} \Delta\Phi &= \int_{S(t+\Delta t)} \vec{B}(t+\Delta t) \cdot d\vec{A} - \int_{S(t)} \vec{B}(t) \cdot d\vec{A} \\ &= \int_{S(t+\Delta t)} \vec{B}(t+\Delta t) \cdot d\vec{A} - \int_{S(t+\Delta t)} \vec{B}(t) \cdot d\vec{A} \\ &\quad + \int_{S(t+\Delta t)} \vec{B}(t) \cdot d\vec{A} - \int_{S(t)} \vec{B}(t) \cdot d\vec{A} \end{aligned} \quad (13.35)$$

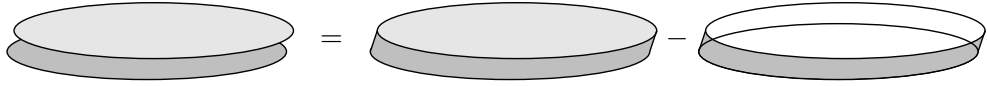
\vec{B} is a function of \vec{r} too, but I won't write it. The first two terms have the same surface, so they combine to give

$$\int_{S(t+\Delta t)} \Delta\vec{B} \cdot d\vec{A}$$

and when you divide by Δt and let it approach zero, you get

$$\int_{S(t)} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Now for the next two terms, which require some manipulation. Add and subtract the surface that forms the edge between the boundaries $C(t)$ and $C(t + \Delta t)$.



$$\int_{S(t+\Delta t)} \vec{B}(t) \cdot d\vec{A} - \int_{S(t)} \vec{B}(t) \cdot d\vec{A} = \oint \vec{B}(t) \cdot d\vec{A} - \int_{\text{edge}} \vec{B} \cdot d\vec{A} \quad (13.36)$$

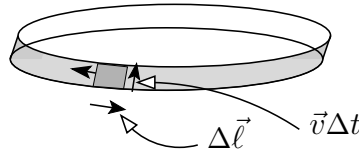
The strip around the edge between the two surfaces make the surface integral closed, but I then have to subtract it as a separate term.

You can convert the surface integral to a volume integral with Gauss's theorem, but it's still necessary to figure out how to write the volume element. [Yes, $\nabla \cdot \vec{B} = 0$, but this result can be applied in other cases too, so don't use that fact here.] The surface is moving at velocity \vec{v} , so an area element $\Delta \vec{A}$ will in time Δt sweep out a volume $\Delta \vec{A} \cdot \vec{v} \Delta t$. Note: \vec{v} isn't necessarily a constant in space and these surfaces aren't necessarily flat.

$$\Delta V = \Delta \vec{A} \cdot \vec{v} \Delta t \implies \oint \vec{B}(t) \cdot d\vec{A} = \int_{S(t)} d^3r \nabla \cdot \vec{B} = \int_{S(t)} \nabla \cdot \vec{B} d\vec{A} \cdot \vec{v} \Delta t \quad (13.37)$$

To do the surface integral around the edge, use the same method as in deriving Stokes' theorem, Eq. (13.20).

$$\Delta \vec{A} = \Delta \vec{\ell} \times \vec{v} \Delta t$$



$$\int_{\text{edge}} \vec{B} \cdot d\vec{A} = \int_C \vec{B} \cdot d\vec{\ell} \times \vec{v} \Delta t = \int_C \vec{v} \times \vec{B} \cdot d\vec{\ell} \Delta t \quad (13.38)$$

Put Eqs. (13.37) and (13.38) into Eq. (13.36) and then into Eq. (13.35), divide by Δt and let $\Delta t \rightarrow 0$.

$$\frac{d}{dt} \int_{S(t)} \vec{B} \cdot d\vec{A} = \int_{S(t)} \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A} + \int_{S(t)} \nabla \cdot \vec{B} \vec{v} \cdot d\vec{A} - \int_{C(t)} \vec{v} \times \vec{B} \cdot d\vec{\ell} \quad (13.39)$$

This transport theorem is the analog of Eq. (13.32) for a surface integral.

In order to check this equation, and to see what the terms do, try some example vector functions that isolate the terms, so that only one of the terms on the right side of (13.39) is non-zero at a time.

1: $\vec{B} = B_0 \hat{z} t$, with a surface $z = 0$, $x^2 + y^2 < R^2$

For a constant B_0 , and $\vec{v} = 0$, only the first term is present. The equation is $B_0 \pi R^2 = B_0 \pi R^2$. Now take a static field

2: $\vec{B} = C z \hat{z}$, with a moving surface $z = vt$, $x^2 + y^2 < R^2$

The first and third terms on the right vanish, and $\nabla \cdot \vec{B} = C$. The other terms are

$$\left. \frac{d}{dt} C z \hat{z} \cdot \pi R^2 \hat{z} \right|_{z=vt} = C v \pi R^2 = \int (C) v \hat{z} \cdot d\vec{A} = C v \pi R^2$$

Now take a uniform static field

$$3: \quad \vec{B} = B_0 \hat{z} \quad \text{with a radially expanding surface} \quad z = 0, \quad x^2 + y^2 < R^2, \quad R = vt$$

The first and second terms on the right are now zero, and

$$\begin{aligned} \frac{d}{dt} B_0 \pi (vt)^2 &= 2B_0 \pi v^2 t = - \oint (v \hat{r} \times B_0 \hat{z}) \cdot \hat{\theta} dl \\ &= - \oint (-v B_0 \hat{\theta}) \cdot \hat{\theta} dl = +v B_0 2\pi R \Big|_{R=vt} = 2B_0 \pi v^2 t \end{aligned}$$

Draw some pictures of these three cases to see if the pictures agree with the algebra.

Faraday's Law

If you now apply the transport theorem (13.39) to Maxwell's equation (13.34), and use the fact that $\nabla \cdot \vec{B} = 0$ you get

$$\int_{C(t)} (\vec{E} + \vec{v} \times \vec{B}) \cdot d\vec{\ell} = - \frac{d}{dt} \int_{S(t)} \vec{B} \cdot d\vec{A} \quad (13.40)$$

This is Faraday's law, saying that the force per charge integrated around a closed loop (called the EMF) is the negative time derivative of the magnetic flux through the loop.

Occasionally you will find an introductory physics text that writes Faraday's law without the $\vec{v} \times \vec{B}$ term. That's o.k. as long as the integrals involve only stationary curves and surfaces, but some will try to apply it to generators, with moving conductors. This results in amazing contortions to try to explain the results. For another of Maxwell's equations, see problem 13.30.

The electromagnetic force on a charge is $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$. This means that if a charge inside a conductor is free to move, the force on it comes from both the electric and the magnetic fields in this equation. (The Lorentz force law.) The integral of this force $\cdot d\vec{\ell}$ is the work done on a charge along some specified path. If this integral is independent of path: $\nabla \times \vec{E} = 0$ and $\vec{v} = 0$, then this work divided by the charge is the potential difference, the voltage, between the initial and final points. In the more general case, where one or the other of these requirements is false, then it's given the somewhat antiquated name EMF, for "electromotive force." (It is often called "voltage" anyway, though if you're being fussy that's not really correct.)

13.6 Fields as Vector Spaces

It's sometimes useful to look back at the general idea of a vector space and to rephrase some common ideas in that language. Vector fields, such as $\vec{E}(x, y, z)$ can be added and multiplied by scalars. They form vector spaces, infinite dimensional of course. They even have a natural scalar product

$$\langle \vec{E}_1, \vec{E}_2 \rangle = \int d^3r \vec{E}_1(\vec{r}) \cdot \vec{E}_2(\vec{r}) \quad (13.41)$$

Here I'm assuming that the scalars are real numbers, though you can change that if you like. For this to make sense, you have to assume that the fields are square integrable, but for the case of electric or magnetic fields that just means that the total energy in the field is finite. Because these are supposed to satisfy some differential equations (Maxwell's), the derivative must also be square integrable, and I'll require that they go to zero at infinity faster than $1/r^3$ or so.

The curl is an operator on this space, taking a vector field into another vector field. Recall the definitions of symmetric and hermitian operators from section 7.14. The curl satisfies the identity

$$\langle \vec{E}_1, \nabla \times \vec{E}_2 \rangle = \langle \nabla \times \vec{E}_1, \vec{E}_2 \rangle \quad (13.42)$$

For a proof, just write it out and then find the vector identity that will allow you to integrate by parts.

$$\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B} \quad (13.43)$$

Equation (13.42) is

$$\int d^3r \vec{E}_1(\vec{r}) \cdot \nabla \times \vec{E}_2(\vec{r}) = \int d^3r (\nabla \times \vec{E}_1(\vec{r})) \cdot \vec{E}_2(\vec{r}) - \int d^3r \nabla \cdot (\vec{E}_1 \times \vec{E}_2)$$

The last integral becomes a surface integral by Gauss's theorem, $\oint d\vec{A} \cdot (\vec{E}_1 \times \vec{E}_2)$, and you can now let the volume (and so the surface) go to infinity. The fields go to zero sufficiently fast, so this is zero and the result is proved: Curl is a symmetric operator. Its eigenvalues are real and its eigenvectors are orthogonal. This is not a result you will use often, but the next one is important.

Helmholtz Decomposition

There are subspaces in this vector space of fields: (1) The set of all fields that are gradients. (2) The set of all fields that are curls. These subspaces are orthogonal to each other; every vector in the first is orthogonal to every vector in the second. To prove this, just use the same vector identity (13.43) and let $\vec{A} = \nabla f$. I will first present a restricted version of this theorem because it's simpler. Assume that the domain is all space and that the fields and their derivatives all go to zero infinitely far away. A generalization to finite boundaries will be mentioned at the end.

$$\nabla f \cdot \nabla \times \vec{B} = \vec{B} \cdot \nabla \times \nabla f - \nabla \cdot (\nabla f \times \vec{B})$$

Calculate the scalar product of one vector field with the other.

$$\begin{aligned} \langle \nabla f, \nabla \times \vec{B} \rangle &= \int d^3r \nabla f \cdot \nabla \times \vec{B} = \int d^3r [\vec{B} \cdot \nabla \times \nabla f - \nabla \cdot (\nabla f \times \vec{B})] \\ &= 0 - \oint (\nabla f \times \vec{B}) \cdot d\vec{A} = 0 \end{aligned} \quad (13.44)$$

As usual, the boundary condition that the fields and their derivatives go to zero rapidly at infinity kills the surface term. This proves the result, that the two subspaces are mutually orthogonal.

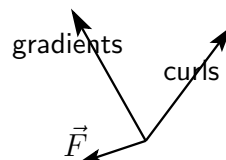
Do these two cases exhaust all possible vector fields? In this restricted case with no boundaries short of infinity, the answer is yes. The general case later will add other possibilities. Here you have two orthogonal subspaces, and to show that these two fill out the whole vector space, I will ask the question: what are all the vector fields orthogonal to *both* of them? I will show first that whatever they are will satisfy Laplace's equation, and then the fact that the fields go to zero at infinity will be enough to show that this third case is identically zero. This statement is the Helmholtz theorem: Such vector fields can be written as the sum of two orthogonal fields: a gradient, and a curl.

To prove it, my plan of attack is to show that if a field \vec{F} is orthogonal to all gradients and to all curls, then $\nabla^2 \vec{F}$ is orthogonal to *all* square-integrable vector fields. The only vector that is orthogonal to everything is the zero vector, so \vec{F} satisfies Laplace's equation. The assumption now is that for general f and \vec{v} ,

$$\int d^3r \vec{F} \cdot \nabla f = 0 \quad \text{and} \quad \int d^3r \vec{F} \cdot \nabla \times \vec{v} = 0$$

I want to show that for a general vector field \vec{u} ,

$$\int d^3r \vec{u} \cdot \nabla^2 \vec{F} = 0$$



The method is essentially two partial integrals, moving two derivatives from \vec{F} over to \vec{u} . Start with the $\partial^2/\partial z^2$ term in the Laplacian and hold off on the dx and dy integrals. Remember that all these functions go to zero at infinity. Pick the i -component of \vec{u} and the j -component of \vec{F} .

$$\begin{aligned} \int_{-\infty}^{\infty} dz u_i \frac{\partial^2}{\partial z^2} F_j &= u_i \partial_z F_j \Big|_{-\infty}^{\infty} - \int dz (\partial_z u_i) (\partial_z F_j) \\ &= 0 - (\partial_z u_i) F_j \Big|_{-\infty}^{\infty} + \int dz (\partial_z^2 u_i) F_j = \int_{-\infty}^{\infty} dz (\partial_z^2 u_i) F_j \end{aligned}$$

Now reinsert the dx and dy integrals. Repeat this for the other two terms in the Laplacian, $\partial_x^2 F_j$ and $\partial_y^2 F_j$. The result is

$$\int d^3r \vec{u} \cdot \nabla^2 \vec{F} = \int d^3r (\nabla^2 \vec{u}) \cdot \vec{F} \quad (13.45)$$

If this looks familiar it is just the three dimensional version of the manipulations that led to Eq. (5.15).

Now use the identity

$$\nabla \times (\nabla \times \vec{u}) = \nabla(\nabla \cdot \vec{u}) - \nabla^2 \vec{u}$$

in the right side of (13.45) to get

$$\int d^3r \vec{u} \cdot \nabla^2 \vec{F} = \int d^3r \left[(\nabla(\nabla \cdot \vec{u})) \cdot \vec{F} - (\nabla \times (\nabla \times \vec{u})) \cdot \vec{F} \right] \quad (13.46)$$

The first term on the right is the scalar product of the vector field \vec{F} with a gradient. The second term is the scalar product with a curl. Both are zero by the hypotheses of the theorem, thereby demonstrating that the Laplacian of \vec{F} is orthogonal to everything, and so $\nabla^2 \vec{F} = 0$.

When you do this in all of space, with the boundary conditions that the fields all go to zero at infinity, the only solutions to Laplace's equation are identically zero. In other words, the two vector spaces (the gradients and the curls) exhaust all the possibilities. How to prove this? Just pick a component, say F_x , treat it as simply a scalar function — call it f — and apply a vector identity, problem 9.36.

$$\nabla \cdot (\phi \vec{A}) = (\nabla \phi) \cdot \vec{A} + \phi (\nabla \cdot \vec{A})$$

$$\text{Let } \phi = f \text{ and } \vec{A} = \nabla f, \quad \text{then} \quad \nabla \cdot (f \nabla f) = \nabla f \cdot \nabla f + f \nabla^2 f$$

Integrate this over all space and apply Gauss's theorem. (I.e. integrate by parts.)

$$\int d^3r \nabla \cdot (f \nabla f) = \oint f \nabla f \cdot d\vec{A} = \int d^3r [\nabla f \cdot \nabla f + f \nabla^2 f] \quad (13.47)$$

If f and its derivative go to zero fast enough at infinity (a modest requirement), the surface term, $\oint f \nabla f \cdot d\vec{A}$, goes to zero. The Laplacian term, $\nabla^2 f = 0$, and all that's left is

$$\int d^3r \nabla f \cdot \nabla f = 0$$

This is the integral of a quantity that can never be negative. The only way that the integral can be zero is that the integrand is zero. If $\nabla f = 0$, then f is a constant, and if it must also go to zero far away then that constant is zero.

This combination of results, the Helmholtz theorem, describes a field as the sum of a gradient and a curl, but is there a way to find these two components explicitly? Yes.

$$\vec{F} = \nabla f + \nabla \times B, \quad \text{so} \quad \nabla \cdot \vec{F} = \nabla^2 f, \quad \text{and} \quad \nabla \times \vec{F} = \nabla \times \nabla \times B = \nabla(\nabla \cdot \vec{B}) - \nabla^2 \vec{B}$$

Solutions of these equations are

$$f(\vec{r}) = \frac{-1}{4\pi} \int d^3r' \frac{\nabla \cdot \vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad \text{and} \quad \vec{B}(\vec{r}) = \frac{1}{4\pi} \int d^3r' \frac{\nabla \times \vec{F}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (13.48)$$

Generalization

In all this derivation, I assumed that the domain is all of three-dimensional space, and this made the calculations easier. A more general result lets you specify boundary conditions on some finite boundary and then a general vector field is the sum of as many as five classes of vector functions. This is the Helmholtz-Hodge decomposition theorem, and it has applications in the more complicated aspects of fluid flow (as if there are any simple ones), even in setting up techniques of numerical analysis for such problems. The details are involved, and I will simply refer you to a good review article* on the subject.

Exercises

1 For a circle, from the definition of the integral, what is $\oint d\vec{\ell}$? What is $\oint d\ell$? What is $\oint d\vec{\ell} \times \vec{C}$ where \vec{C} is a constant vector?

2 What is the work you must do in lifting a mass m in the Earth's gravitational field from a radius R_1 to a radius R_2 . These are measured from the center of the Earth and the motion is purely radial.

3 Same as the preceding exercise but the motion is 1. due north a distance $R_1\theta_0$ then 2. radially out to R_2 then 3. due south a distance $R_2\theta_0$.

4 Verify Stokes' Theorem by separately calculating the left and the right sides of the theorem for the case of the vector field

$$\vec{F}(x, y) = \hat{x} Ay + \hat{y} Bx$$

around the rectangle $(a < x < b)$, $(c < y < d)$.

5 Verify Stokes' Theorem by separately calculating the left and the right sides of the theorem for the case of the vector field

$$\vec{F}(x, y) = \hat{x} Ay - \hat{y} Bx$$

around the rectangle $(a < x < b)$, $(c < y < d)$.

6 Verify Stokes' Theorem for the semi-cylinder $0 < z < h$, $0 < \phi < \pi$, $r = R$. The vector field is $\vec{F}(r, \phi, z) = \hat{r} Ar^2 \sin \phi + \hat{\phi} Br\phi^2 z + \hat{z} Crz^2$

7 Verify Gauss's Theorem using the whole cylinder $0 < z < h$, $r = R$ and the vector field $\vec{F}(r, \phi, z) = \hat{r} Ar^2 \sin \phi + \hat{\phi} Brz \sin^2 \phi + \hat{z} Crz^2$.

8 What would happen if you used the volume of the preceding exercise and the field of the exercise before that one to check Gauss's law?

* Cantarella, DeTurck, and Gluck: The American Mathematical Monthly, May 2002. The paper is an unusual mix of abstract topological methods and very concrete examples. It thereby gives you a fighting chance at the subject.

Problems

13.1 In the equation (13.4) what happens if you start with a different parametrization for x and y , perhaps $x = R \cos(\phi'/2)$ and $y = R \sin(\phi'/2)$ for $0 < \phi' < 4\pi$. Do you get the same answer?

13.2 What is the length of the arc of the parabola $y = (a^2 - x^2)/b$, $(-a < x < a)$?

But First draw a sketch and make a rough estimate of what the result ought to be. *Then* do the calculation and compare the answers. What limiting cases allow you to check your result?

Ans: $(b/2)[\sinh^{-1} c + c\sqrt{1+c^2}]$ where $c = 2a/b$

13.3 (a) You can describe an ellipse as $x = a \cos \phi$, $y = b \sin \phi$. (Prove this.)

(b) Warm up by computing the area of the ellipse.

(c) What is the circumference of this ellipse? You will find a (probably) unfamiliar integral here, so to put this integral into a standard form, note that it is $4 \int_0^{\pi/2}$. Then use $\cos^2 \phi = 1 - \sin^2 \phi$. Finally, look up chapter 17, Elliptic Integrals, of **Abramowitz and Stegun**. You will find the reference to this at the end of section 1.4. Notice in this integral that when you integrate, it will not matter whether you have a \sin^2 or a \cos^2 . Ans: $4aE(m)$

13.4 For another derivation of the work-energy theorem, one that doesn't use the manipulations of calculus as in Eq. (13.11), go back to basics.

(a) For a constant force, start from $\vec{F} = m\vec{a}$ and derive by elementary manipulations that

$$\vec{F} \cdot \Delta \vec{r} = \frac{m}{2} [v_f^2 - v_i^2]$$

All that you need to do is to note that the acceleration is a constant so you can get \vec{v} and \vec{r} as functions of time. Then eliminate t

(b) Along a specified curve Divide the curve at points

$$\vec{r}_i = \vec{r}_0, \vec{r}_1, \vec{r}_2, \dots, \vec{r}_N = \vec{r}_f$$

In each of these intervals apply the preceding equation. This makes sense in that if the interval is small the force won't change much in the interval.

(c) Add all these N equations and watch the kinetic energy terms telescope and (mostly) cancel. This limit as all the $\Delta \vec{r}_k \rightarrow 0$ is Eq. (13.12).

13.5 The manipulation in the final step of Eq. (13.12) seems almost *too* obvious. Is it? Well yes, but write out the definition of this integral as the limit of a sum to verify that it really is easy.

13.6 Mimic the derivation of Gauss's theorem, Eq. (13.15), and derive the identities

$$\oint_S d\vec{A} \times \vec{v} = \int_V \text{curl } \vec{v} dV, \quad \text{and} \quad \oint_S f d\vec{A} = \int_V \text{grad } f dV$$

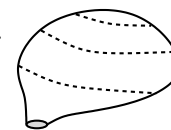
13.7 The force by a magnetic field on a small piece of wire, length $d\ell$, and carrying a current I is $d\vec{F} = I d\vec{\ell} \times \vec{B}$. The total force on a wire carrying this current in a complete circuit is the integral of this. Let $\vec{B} = \hat{x}Ay - \hat{y}Ax$. The wire consists of the line segments around the rectangle $0 < x < a$, $0 < y < b$. The direction of the current is in the $+\hat{y}$ direction on the $x = 0$ line. What is the total force on the loop? Ans: 0

13.8 Verify Stokes' theorem for the field $\vec{F} = Axy\hat{x} + B(1 + x^2y^2)\hat{y}$ and for the rectangular loop $a < x < b$, $c < y < d$.

13.9 Which of the two times in Eqs. (13.7) and (13.8) is shorter. (Compare their squares; it's easier.)

13.10 Write the equations (9.36) in an integral form.

13.11 Start with Stokes' theorem and shrink the boundary curve to a point. That doesn't mean there's no surface left; it's not flat, remember. The surface is pinched off like a balloon. It is now a closed surface, and what is the value of this integral? Now apply Gauss's theorem to it and what do you get? Ans: See Eq. (9.34)



13.12 Use the same surface as in the example, Eq. (13.25), and verify Stokes' theorem for the vector field

$$\vec{F} = \hat{r} Ar^{-1} \cos^2 \theta \sin \phi + \hat{\theta} Br^2 \sin \theta \cos^2 \phi + \hat{\phi} Cr^{-2} \cos^2 \theta \sin^2 \phi$$

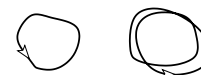
13.13 Use the same surface as in the example, Eq. (13.25), and examine Stokes' theorem for the vector field

$$\vec{F} = \hat{r} f(r, \theta, \phi) + \hat{\theta} g(r, \theta, \phi) + \hat{\phi} h(r, \theta, \phi)$$

(a) Show from the line integral part that the answer can depend only on the function h , not f or g .

(b) Now examine the surface integral over this cap and show the same thing.

13.14 For the vector field in the x - y plane: $\vec{F} = (x\hat{y} - y\hat{x})/2$, use Stokes' theorem to compute the line integral of $\vec{F} \cdot d\vec{r}$ around an arbitrary closed curve. What is the significance of the sign of the result? When you considered an "arbitrary" loop, did you consider the possibilities presented by these curves?



13.15 What is the (closed) surface integral of $\vec{F} = \vec{r}/3$ over an arbitrary closed surface? Ans: V .

13.16 What is the (closed) surface integral of $\vec{F} = \vec{r}/3$ over an arbitrary closed surface? This time however, the surface integral uses the cross product: $\oint d\vec{A} \times \vec{F}$. If in doubt, try drawing the picture for a special case first.

13.17 For the vector field Eq. (13.28) explicitly show that $\oint \vec{v} \cdot d\vec{r}$ is zero for a curve such as that in the picture and that it is not zero for a circle going around the singularity.



13.18 Refer to Eq. (13.27) and check it for small θ_0 . Notice the combination $\pi(R\theta_0)^2$.



13.19 For the vector field, Eq. (13.28), use Eq. (13.29) to try to construct a potential function. Because within a certain domain the integral is independent of path, you can pick the most convenient possible path, the one that makes the integration easiest. What goes wrong?

13.20 Refer to problem 9.33 and construct the solutions by integration, using the methods of this chapter.

13.21 (a) Evaluate $\oint \vec{F} \cdot d\vec{r}$ for $\vec{F} = \hat{x}Axy + \hat{y}Bx$ around the circle of radius R centered at the origin.

(b) Now do it again, using Stokes' theorem this time.

13.22 Same as the preceding problem, but $\oint d\vec{r} \times \vec{F}$ instead.

13.23 Use the same field as the preceding two problems and evaluate the surface integral of $\vec{F} \cdot d\vec{A}$ over the hemispherical surface $x^2 + y^2 + z^2 = R^2$, $z > 0$.

13.24 The same field and surface as the preceding problem, but now the surface integral $d\vec{A} \times \vec{F}$.
Ans: $\hat{z} 2\pi B r^3 / 3$

13.25 (a) Prove the identity $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B}$. (index mechanics?)

(b) Next apply Gauss's theorem to $\nabla \cdot (\vec{A} \times \vec{B})$ and take the special case that \vec{B} is an arbitrary constant to derive Eq. (13.19).

13.26 (a) Prove the identity $\nabla \cdot (f\vec{F}) = f\nabla \cdot \vec{F} + \vec{F} \cdot \nabla f$.

(b) Apply Gauss's theorem to $\nabla \cdot (f\vec{F})$ for an arbitrary constant \vec{F} to derive a result found in another problem.

(c) Explain why the word "arbitrary" is necessary here.

13.27 The vector potential is not unique, as you can add an arbitrary gradient to it without affecting its curl. Suppose that $\vec{B} = \nabla \times \vec{A}$ with

$$\vec{A} = \hat{x} \alpha xyz + \hat{y} \beta x^2 z + \hat{z} \gamma xyz^2$$

Find a function $f(x, y, z)$ such that $\vec{A}' = \vec{A} + \nabla f$ has the z -component identically zero. Do you get the same \vec{B} by taking the curl of \vec{A} and of \vec{A}' ?

13.28 Take the vector field

$$\vec{B} = \alpha xy \hat{x} + \beta xy \hat{y} + \gamma(xz + yz) \hat{z}$$

Write out the equation $\vec{B} = \nabla \times \vec{A}$ in rectangular components and figure out what functions $A_x(x, y, z)$, $A_y(x, y, z)$, and $A_z(x, y, z)$ will work. Note: From the preceding problem you see that you may if you wish pick any one of the components of \vec{A} to be zero — that will cut down on the labor. Also, you should expect that this problem is impossible unless \vec{B} has zero divergence. That fact should come out of your calculations, so don't put it in yet. Determine the conditions on α , β , and γ that make this problem solvable, and show that this is equivalent to $\nabla \cdot \vec{B} = 0$.

13.29 A magnetic monopole, if it exists, will have a magnetic field $\mu_0 q_m \hat{r} / 4\pi r^2$. The divergence of this magnetic field is zero except at the origin, but that means that not every closed surface can be shrunk to a point without running into the singularity. The necessary condition for having a vector potential is not satisfied. Try to construct such a potential anyway. Assume a solution in spherical coordinates of the form $\vec{A} = \hat{\phi} f(r)g(\theta)$ and figure out what f and g will have this \vec{B} for a curl. Sketch the resulting \vec{A} . You will run into a singularity (or two, depending). Ans: $\vec{A} = \hat{\phi} \mu_0 q_m (1 - \cos \theta) / (4\pi r^2 \sin \theta)$ (not unique)

13.30 Apply the Reynolds transport theorem to the other of Maxwell's equations.

$$\nabla \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Don't simply leave the result in the first form that you find. Manipulate it into what seems to be the best form. Use $\mu_0 \epsilon_0 = 1/c^2$.

Ans: $\oint (\vec{B} - \vec{v} \times \vec{E}/c^2) \cdot d\vec{\ell} = \mu_0 \int (\vec{j} - \rho \vec{v}) \cdot d\vec{A} + \mu_0 \epsilon_0 (d/dt) \int \vec{E} \cdot d\vec{A}$

13.31 Derive the analog of the Reynolds transport theorem, Eq. (13.39), for a line integral around a closed loop.

$$(a) \quad \frac{d}{dt} \oint_{C(t)} \vec{F}(\vec{r}, t) \cdot d\vec{\ell} = \oint_{C(t)} \frac{\partial \vec{F}}{\partial t} \cdot d\vec{\ell} - \oint_{C(t)} \vec{v} \times (\nabla \times \vec{F}) \cdot d\vec{\ell}$$

and for the surface integral of a scalar. You will need problem 13.6.

$$(b) \quad \frac{d}{dt} \int_{S(t)} \phi(\vec{r}, t) d\vec{A} = \int_{S(t)} \frac{\partial \phi}{\partial t} d\vec{A} + \int_{S(t)} (\nabla \phi) d\vec{A} \cdot \vec{v} - \oint_{C(t)} \phi d\vec{\ell} \times \vec{v}$$

Make up examples that test the validity of individual terms in the equations. I recommend cylindrical coordinates for your examples.

13.32 Another transport theorem is more difficult to derive.

$$\frac{d}{dt} \oint_{C(t)} d\vec{\ell} \times \vec{F}(\vec{r}, t) = \oint_{C(t)} d\vec{\ell} \times \frac{\partial \vec{F}}{\partial t} + \oint_{C(t)} (\nabla \cdot \vec{F}) d\vec{\ell} \times \vec{v} - \int_{C(t)} (\nabla \vec{F}) \cdot d\vec{\ell} \times \vec{v}$$

I had to look up some vector identities, including one for $\nabla \times (\vec{A} \times \vec{B})$. A trick that I found helpful: At a certain point take the dot product of the whole equation with a fixed vector \vec{B} and manipulate the resulting product, finally factoring out the arbitrary vector \vec{B} at the end. Make up examples that test the validity of individual terms in the equations. Again, I recommend cylindrical coordinates for your examples.

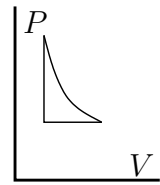
13.33 Apply Eq. (13.39) to the velocity field itself. That is, let $\vec{B} = \vec{v}$. Suppose further that the fluid is incompressible with $\nabla \cdot \vec{v} = 0$ and that the flow is stationary (no time dependence). Explain the results.

13.34 Assume that the Earth's atmosphere obeys the density equation $\rho = \rho_0 e^{-z/h}$ for a height z above the surface. (a) Through what amount of air does sunlight have to travel when coming from straight overhead? Take the measure of this to be $\int \rho d\ell$ (called the "air mass"). (b) Through what amount of air does sunlight have to travel when coming from just on the horizon at sunset? Neglect the fact that light will refract in the atmosphere and that the path in the second case won't really be a straight line. Take $h = 10$ km and the radius of the Earth to be 6400 km. The integral you get for the second case is probably not familiar. You may evaluate it numerically for the numbers that I stated, or you may look it up in a big table of integrals such as Gradshteyn and Ryzhik, or you may use an approximation, $h \ll R$. (I recommend the last.) What is the numerical value of the ratio of these two air mass integrals? This goes far in explaining why you can look at the setting sun.

(c) If refraction in the atmosphere is included, the light will bend and pass through a still larger air mass. The overall refraction comes to about 0.5° , and calculating the path that light takes is hard, but you can find a bound on the answer by assuming a path that follows the surface of the Earth through this angle and then takes off on a straight line. What is the air mass ratio in this case? The real answer is somewhere between the two calculations. (The *really* real answer is a little bigger than either because the atmosphere is not isothermal and so the approximation $\rho = \rho_0 e^{-z/h}$ is not exact.)

Ans: $\approx \sqrt{R\pi/2h} = 32, \quad +R\theta/h \rightarrow 37.$

13.35 Work in a thermodynamic system is calculated from $dW = P dV$. Assume an ideal gas, so that $PV = nRT$. (a) What is the total work, $\oint dW$, done around this cycle as the pressure increases at constant volume, then decreases at constant temperature, finally the volume decreases at constant pressure.



(b) In the special case for which the changes in volume and pressure are very small, estimate from the graph approximately what to expect for the answer. Now do an expansion of the result of part (a) to see if it agrees with what you expect. Ans: $\approx \Delta P \Delta V/2$

13.36 Verify the divergence theorem for the vector field

$$\vec{F} = \alpha xyz \hat{x} + \beta x^2 z(1+y) \hat{y} + \gamma xyz^2 \hat{z}$$

and for the volume $(0 < x < a)$, $(0 < y < b)$, $(0 < z < c)$.

13.37 Evaluate $\int \vec{F} \cdot d\vec{A}$ over the curved surface of the hemisphere $x^2 + y^2 + z^2 = R^2$ and $z > 0$. The vector field is given by $\vec{F} = \nabla \times (\alpha y \hat{x} + \beta x \hat{y} + \gamma xy \hat{z})$. Ans: $(\beta - \alpha)\pi R^2$

13.38 A vector field is given in cylindrical coordinates to be $\vec{F} = \hat{r} \alpha r^2 z \sin^2 \phi + \hat{\phi} \beta r z + \hat{z} \gamma z r \cos^2 \phi$. Verify the divergence theorem for this field for the region $(0 < r < R)$, $(0 < \phi < 2\pi)$, $(0 < z < h)$.

13.39 For the function $F(r, \theta) = r^n(A + B \cos \theta + C \cos^2 \theta)$, compute the gradient and then the divergence of this gradient. For what values of the constants A , B , C , and (positive, negative, or zero) integer n is this result, $\nabla \cdot \nabla F$, zero? These coordinates are spherical, and this combination div grad is the Laplacian.

Ans: In part, $n = 2$, $C = -3A$, $B = 0$.

13.40 Repeat the preceding problem, but now interpret the coordinates as cylindrical (change θ to ϕ). And don't necessarily leave your answers in the first form that you find them.

13.41 Evaluate the integral $\int \vec{F} \cdot d\vec{A}$ over the surface of the hemisphere $x^2 + y^2 + z^2 = 1$ with $z > 0$. The vector field is $\vec{F} = A(1+x+y)\hat{x} + B(1+y+z)\hat{y} + C(1+z+x)\hat{z}$. You may choose to do this problem the hard way or the easy way. Or both.

Ans: $\pi(2A + 2B + 5C)/3$

13.42 An electric field is known in cylindrical coordinates to be $\vec{E} = f(r)\hat{r}$, and the electric charge density is a function of r alone, $\rho(r)$. They satisfy the Maxwell equation $\nabla \cdot \vec{E} = \rho/\epsilon_0$. If the charge density is given as $\rho(r) = \rho_0 e^{-r/r_0}$. Compute \vec{E} . Demonstrate the behavior of \vec{E} is for large r and for small r .

13.43 Repeat the preceding problem, but now r is a spherical coordinate.

13.44 Find a vector field \vec{F} such that $\nabla \cdot \vec{F} = \alpha x + \beta y + \gamma$ and $\nabla \times \vec{F} = \hat{z}$. Next, find an infinite number of such fields.

13.45 Gauss's law says that the total charge contained inside a surface is $\epsilon_0 \oint \vec{E} \cdot d\vec{A}$. For the electric field of problem 10.37, evaluate this integral over a sphere of radius $R_1 > R$ and centered at the origin.

13.46 (a) In cylindrical coordinates, for what n does the vector field $\vec{v} = r^n \hat{\phi}$ have curl equal to zero? Draw it.

(b) Also, for the same closed path as in problem 13.17 and for all n , compute $\oint \vec{v} \cdot d\vec{r}$.

13.47 Prove the identity Eq. (13.43). Write it out in index notation first.

13.48 There is an analog of Stokes' theorem for $\oint d\vec{\ell} \times \vec{B}$. This sort of integral comes up in computing the total force on the current in a circuit. Try multiplying (dot) the integral by a constant vector \vec{C} . Then manipulate the result by standard methods and hope that in the end you have the same constant $\vec{C} \cdot$ something.

Ans: $= \int [(\nabla \vec{B}) \cdot d\vec{A} - (\nabla \cdot \vec{B}) \cdot d\vec{A}]$ and the second term vanishes for magnetic fields.

13.49 In the example (13.16) using Gauss's theorem, the term in γ contributed zero to the surface integral (13.17). In the subsequent volume integral the same term vanished because of the properties of $\sin \phi \cos \phi$. *But* this term will vanish in the surface integral no matter what the function of ϕ is in the $\hat{\phi}$ component of the vector \vec{F} . How then is it always guaranteed to vanish in the volume integral?

13.50 Interpret the vector field \vec{F} from problem 13.37 as an electric field \vec{E} , then use Gauss's law that $q_{\text{enclosed}} = \epsilon_0 \oint \vec{E} \cdot d\vec{A}$ to evaluate the charge enclosed within a sphere of radius R centered at the origin.

13.51 Derive the identity Eq. (13.32) starting from the definition of a derivative and doing the same sort of manipulation that you use in deriving the ordinary product rule for differentiation.

13.52 A right tetrahedron has three right triangular sides that meet in one vertex. Think of a corner chopped off of a cube. The sum of the squares of the areas of the three right triangles equals the square of the area of the fourth face. The results of problem 13.6 will be useful.

