Vector Calculus 1

The first rule in understanding vector calculus is *draw lots of pictures.* This subject can become rather abstract if you let it, but try to visualize all the manipulations. Try a lot of special cases and explore them. Keep relating the manipulations to the underlying pictures and don't get lost in the forest of infinite series. Along with the pictures, there are three types of derivatives, a couple of types of integrals, and some theorems relating them.

9.1 Fluid Flow

When water or any fluid moves through a pipe, what is the relationship between the motion of the fluid and the total rate of flow through the pipe (volume per time)? Take a rectangular pipe of sides a and b with fluid moving at constant speed through it and with the velocity of the fluid being the same throughout the pipe. It's a simple calculation: In time Δt the fluid moves a distance $v\Delta t$ down the pipe. The cross-section of the pipe has area $A = ab$, so the volume that move past a given flat surface is $\Delta V = Av\Delta t$. The flow rate is the volume per time, $\Delta V/\Delta t = Av$. (The usual limit as $\Delta t \to 0$ isn't needed here.)

Just to make the problem look a little more involved, what happens to the result if I ask for the flow through a surface that is tilted at an angle to the velocity. Do the calculation the same way as before, but use the drawing (b) instead of (a). The fluid still moves a distance $v\Delta t$, but the volume that moves past this flat but tilted surface is not its new (bigger) area A times $v\Delta t$. The area of a parallelogram is not the product of its sides and the volume of a parallelepiped is not the area of a base times the length of another side.

The area of a parallelogram is the length of one side times the perpendicular distance from that side to its opposite side. Similarly the volume of a parallelepiped is the area of one side times the perpendicular distance from that side to the side opposite. The perpendicular distance is not the distance that the fluid moved ($v\Delta t$). This perpendicular distance is smaller by a factor $\cos\alpha$, where α is the angle that the plane is tilted. It is most easily described by the angle that the *normal* to the plane makes with the direction of the fluid velocity.

$$\Delta V = Ah = A(v\Delta t)\cos\alpha$$

The flow rate is then $\Delta V/\Delta t = Av\cos\alpha$. Introduce the unit normal vector $\hat{n}$, then this expression can be rewritten in terms of a dot product,

$$Av\cos\alpha = A\vec{v}\cdot\hat{n} = \vec{A}\cdot\vec{v} \tag{9.1}$$

where α is the angle between the direction of the fluid velocity and the normal to the area.

This invites the definition of the area itself as a vector, and that's what I wrote in the final expression. The vector $\vec{A}$ is a notation for $A\hat{n}$, and defines the area vector. If it looks a little odd to have an area be a vector, do you recall the geometric interpretation of a cross product? That's the vector perpendicular to two given vectors and it has a magnitude equal to the area of the parallelogram between the two vectors. It's the same thing.
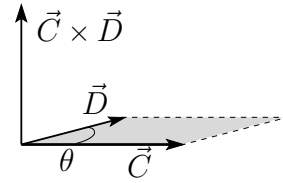
General Flow, Curved Surfaces

The fluid velocity will not usually be a constant in space. It will be some function of position. The surface doesn't have to be flat; it can be cylindrical or spherical or something more complicated. How do you handle this? That's why integrals were invented.

The idea behind an integral is that you will divide a complicated thing into small pieces and add the results of the small pieces to *estimate* the whole. Improve the estimation by making more, smaller pieces, and in the limit as the size of the pieces goes to zero get an exact answer. That's the procedure to use here.

The concept of the surface integral is that you systematically divide a surface into a number (N) of pieces $(k = 1, 2, \ldots N)$. The pieces have area ΔA_k and each piece has a unit normal vector $\hat{n}_k$. Within the middle of each of these areas the fluid has a velocity $\vec{v}_k$. This may not be a constant, but as usual with integrals, you pick a point somewhere in the little area and pick the $\vec{v}$ there; in the limit as all the pieces of area shrink to zero it won't matter exactly where you picked it. The flow rate through one of these pieces is Eq. (9.1), $\vec{v}_k \cdot \hat{n}_k \Delta A_k$, and the corresponding estimate of the total flow through the surface is, using the notation $\Delta \vec{A}_k = \hat{n}_k \Delta A_k$,

$$\sum_{k=1}^{N} \vec{v}_k \cdot \Delta \vec{A}_k$$

This limit as the size of each piece is shrunk to zero and correspondingly the number of pieces goes to infinity is the definition of the integral

$$\int \vec{v} \cdot d\vec{A} = \lim_{\Delta A_k \to 0} \sum_{k=1}^{N} \vec{v}_k \cdot \Delta \vec{A}_k \qquad (9.2)$$

Example of Flow Calculation

In the rectangular pipe above, suppose that the flow exhibits shear, rising from zero at the bottom to v_0 at the top. The velocity field is

$$\vec{v}(x, y, z) = v_x(y)\,\hat{x} = v_0 \frac{y}{b}\hat{x} \qquad (9.3)$$

The origin is at the bottom of the pipe and the y-coordinate is measured upward from the origin. What is the flow rate through the area indicated, tilted at an angle ϕ from the vertical? The distance in and out of the plane of the picture (the z-axis) is the length a. Can such a fluid flow really happen? Yes, real fluids such as water have viscosity, and if you construct a very wide pipe but not too high, and leave the top open to the wind, the horizontal wind will drag the fluid at the top with it (even if not as fast). The fluid at the bottom is kept at rest by the friction with the bottom surface. In between you get a gradual transition in the flow that is represented by Eq. (9.3).
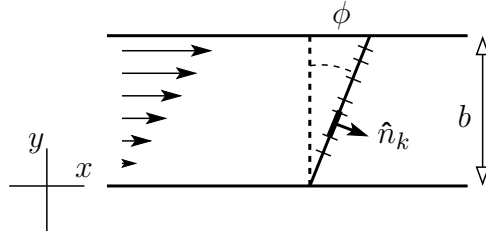
Now to implement the calculation of the flow rate:

Divide the area into N pieces of length $\Delta\ell_k$ along the slant.

The length in and out is a so the piece of area is $\Delta A_k = a\Delta\ell_k$.

The unit normal is $\hat{n}_k = \hat{x}\cos\phi - \hat{y}\sin\phi$. (It happens to be independent of the index k, but that's special to this example.)

The velocity vector at the position of this area is $\vec{v} = v_0\hat{x}\,y_k/b$.

Put these together and you have the piece of the flow rate contributed by this area.

$$\Delta\text{flow}_k = \vec{v}\cdot\Delta\vec{A}_k = v_0\frac{y_k}{b}\hat{x}\cdot a\Delta\ell_k\big(\hat{x}\cos\phi - \hat{y}\sin\phi\big)$$

$$= v_0\frac{y_k}{b}a\Delta\ell_k\cos\phi = v_0\frac{\ell_k\cos\phi}{b}a\Delta\ell_k\cos\phi$$

In the last line I put all the variables in terms of ℓ, using $y = \ell\cos\phi$.

Now sum over all these areas and take a limit.

$$\lim_{\Delta\ell_k\to 0}\sum_{k=1}^{N}v_0\frac{\ell_k\cos\phi}{b}a\Delta\ell_k\cos\phi = \int_0^{b/\cos\phi}d\ell\,v_0\frac{a}{b}\ell\cos^2\phi = v_0\frac{a}{b}\frac{\ell^2}{2}\cos^2\phi\bigg|_0^{b/\cos\phi}$$

$$= v_0\frac{a}{2b}\left(\frac{b}{\cos\phi}\right)^2\cos^2\phi = v_0\frac{ab}{2}$$

This turns out to be independent of the orientation of the plane; the parameter ϕ is absent from the result. If you think of two planes, at angles ϕ_1 and ϕ_2, what flows into one flows out of the other. Nothing is lost in between.

Another Flow Calculation

Take the same sort of fluid flow in a pipe, but make it a little more complicated. Instead of a flat surface, make it a cylinder. The axis of the cylinder is in and out in the picture and its radius is half the width of the pipe. Describe the coordinates on the surface by the angle θ as measured from the midline. That means that $-\pi/2 < \theta < \pi/2$. Divide the surface into pieces that are rectangular strips of length a (in and out in the picture) and width $b\Delta\theta_k/2$. (The radius of the cylinder is $b/2$.)

$$\Delta A_k = a\frac{b}{2}\Delta\theta_k, \qquad \text{and} \qquad \hat{n}_k = \hat{x}\cos\theta_k + \hat{y}\sin\theta_k \tag{9.4}$$
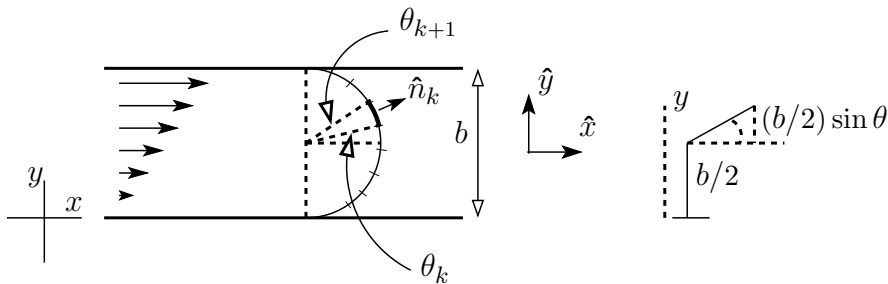
The velocity field is the same as before, $\vec{v}(x,y,z) = v_0\hat{x}\,y/b$, so the contribution to the flow rate

through this piece of the surface is

$$\vec{v}_k \cdot \Delta \vec{A}_k = v_0 \frac{y_k}{b} \hat{x} \cdot a \frac{b}{2} \Delta \theta_k \, \hat{n}_k$$

The value of y_k at the angle θ_k is

$$y_k = \frac{b}{2} + \frac{b}{2} \sin \theta_k, \qquad \text{so} \qquad \frac{y_k}{b} = \frac{1}{2}[1 + \sin \theta_k]$$

Put the pieces together and you have

$$v_0 \frac{1}{2}[1 + \sin \theta_k] \hat{x} \cdot a \frac{b}{2} \Delta \theta_k [\hat{x} \cos \theta_k + \hat{y} \sin \theta_k] = v_0 \frac{1}{2}[1 + \sin \theta_k] a \frac{b}{2} \Delta \theta_k \cos \theta_k$$

The total flow is the sum of these over k and then the limit as $\Delta \theta_k \to 0$.

$$\lim_{\Delta \theta_k \to 0} \sum_k v_0 \frac{1}{2}[1 + \sin \theta_k] a \frac{b}{2} \Delta \theta_k \cos \theta_k = \int_{-\pi/2}^{\pi/2} v_0 \frac{1}{2}[1 + \sin \theta] a \frac{b}{2} \, d\theta \, \cos \theta$$

Finally you can do the two terms of the integral: Look at the second term first. You can of course start grinding away and find the right trigonometric formula to do the integral, OR, you can sketch a graph of the integrand, $\sin \theta \cos \theta$, on the interval $-\pi/2 < \theta < \pi/2$ and write the answer down by inspection. The first part of the integral is

$$v_0 \frac{ab}{4} \int_{-\pi/2}^{\pi/2} \cos \theta = v_0 \frac{ab}{4} \sin \theta \Big|_{-\pi/2}^{\pi/2} = v_0 \frac{ab}{2}$$

And this is the same result as for the flat surface calculation. I set it up so that the two results are the same; it's easier to check that way. Gauss's theorem of vector calculus will guarantee that you get the same result for any surface spanning this pipe *and* for this particular velocity function.

9.2 Vector Derivatives

I want to show the underlying ideas of the vector derivatives, divergence and curl, and as the names themselves come from the study of fluid flow, that's where I'll start. You can describe the flow of a fluid, either gas or liquid or anything else, by specifying its velocity field, $\vec{v}(x, y, z) = \vec{v}(\vec{r})$.
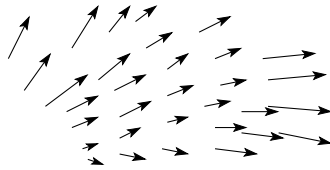
For a single real-valued function of a real variable, it is often too complex to capture all the properties of a function at one glance, so it's going to be even harder here. One of the uses of ordinary calculus is to provide information about the *local* properties of a function without attacking the whole function at once. That is what derivatives do. If you know that the derivative of a function is positive at a point then you know that it is increasing there. This is such an ordinary use of calculus that you hardly give it a second thought (until you hit some advanced calculus and discover that some continuous functions don't even *have* derivatives). The geometric concept of derivative is the slope of the curve at a point — the tangent of the angle between the x-axis and the straight line that best approximates the curve at that point. Going from this geometric idea to calculating the derivative takes some effort.

How can you do this for fluid flow? If you inject a small amount of dye into the fluid at some point it will spread into a volume that depends on how much you inject. As time goes on this region will move and distort and possibly become very complicated, too complicated to grasp in one picture.
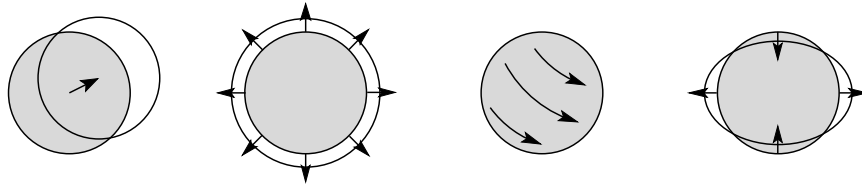
There must be a way to get a simpler picture. There is. Do it in the same spirit that you introduce the derivative, and concentrate on a little piece of the picture. Inject just a little bit of dye and wait only a little time. To make it explicit, assume that the initial volume of dye forms a sphere of (small) volume V and let the fluid move for a little time.

1. In a small time Δt the center of the sphere will move.
2. The sphere can expand or contract, changing its volume.
3. The sphere can rotate.
4. The sphere can distort.

Div, Curl, Strain
The first one, the motion of the center, tells you about the velocity at the center of the sphere. It's like knowing the value of a function at a point, and that tells you nothing about the behavior of the function in the neighborhood of the point.

The second one, the volume, gives new information. You can simply take the time derivative dV/dt to see if the fluid is expanding or contracting; just check the sign and determine if it's positive or negative. But how big is it? That's not yet in a useful form because the size of this derivative will depend on how much the original volume is. If you put in twice as much dye, each part of the volume will change and there will be twice as much rate of change in the total volume. Divide the time derivative by the volume itself and this effect will cancel. Finally, to get the effect at one point take the limit as the volume approaches a point. This defines a kind of derivative of the velocity field called the divergence.

$$\lim_{V \to 0} \frac{1}{V} \frac{dV}{dt} = \text{divergence of } \vec{v} \tag{9.5}$$

This doesn't tell you how to compute it any more than saying that the derivative is the slope tells you how to compute an ordinary* derivative. I'll have to work that out.

But first look at the third way that the sphere can move: rotation. Again, if you take a large object it will distort a lot and it will be hard to define a single rotation for it. Take a very small sphere instead. The time derivative of this rotation is its angular velocity, the vector $\vec{\omega}$. In the limit as the sphere approaches a point, this tells me about the rotation of the fluid in the immediate neighborhood of that point. If I place a tiny paddlewheel in the fluid, how will it rotate?

$$2\vec{\omega} = \text{curl of } \vec{v} \tag{9.6}$$

The factor of 2 is for later convenience.

After considering expansion and rotation, the final way that the sphere can change is that it can alter its shape. In a very small time interval, the sphere can slightly distort into an ellipsoid. This will lead to the mathematical concept of the *strain.* This is important in the subject of elasticity and

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* Can you start from the definition of the derivative as a slope, use it directly with no limits, and compute the derivative of x^2 with respect to x, getting $2x$? It *can* be done.

viscosity, but I'll put it aside for now save for one observation: how much information is needed to describe whatever it is? The sphere changes to an ellipsoid, and the first question is: what is the longest axis and how much stretch occurs along it — that's the three components of a vector. After that what is the shortest axis and how much contraction occurs along *it*? That's one more vector, but you need only two new components to define its direction because it's perpendicular to the long axis. After this there's nothing left. The direction of the third axis is determined and so is its length if you assume that the total volume hasn't changed. You can assume that is so because the question of volume change is already handled by the divergence; you don't need it here too. The total number of components needed for this object is $2 + 3 = 5$. It comes under the heading of tensors.

9.3 Computing the divergence

Now how do you calculate these? I'll start with the simplest, the divergence, and compute the time derivative of a volume from the velocity field. To do this, go back to the definition of derivative:

$$\frac{dV}{dt} = \lim_{\Delta t \to 0} \frac{V(t + \Delta t) - V(t)}{\Delta t} \tag{9.7}$$

$$\vec{v}\Delta t$$
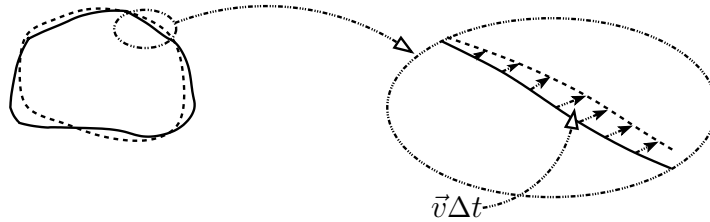
Pick an arbitrary surface to start with and see how the volume changes as the fluid moves, carrying the surface with it. In time Δt a point on the surface will move by a distance $\vec{v}\Delta t$ and it will carry with it a piece of neighboring area ΔA. This area sweeps out a volume. This piece of volume is not ΔA times $v\Delta t$ because the motion isn't likely to be perpendicular to the surface. It's only the *component* of the velocity normal to the surface that contributes to the volume swept out. Use $\hat{n}$ to denote the unit vector perpendicular to ΔA, then this volume is $\Delta A \hat{n} \cdot \vec{v} \Delta t$. This is the same as the calculation for fluid flow except that I'm interpreting the picture differently.

If at a particular point on the surface the normal $\hat{n}$ is more or less in the direction of the velocity then this dot product is positive and the change in volume is positive. If it's opposite the velocity then the change is negative. The total change in volume of the whole initial volume is the sum over the entire surface of all these changes. Divide the surface into a lot of pieces ΔA_i with accompanying unit normals $\hat{n}_i$, then

$$\Delta V_{\text{total}} = \sum_i \Delta A_i \, \hat{n}_i \cdot \vec{v}_i \, \Delta t$$

Not really. I have to take a limit before this becomes an equality. The limit of this as all the $\Delta A_i \to 0$ defines an integral

$$\Delta V_{\text{total}} = \oint dA \, \hat{n} \cdot \vec{v} \, \Delta t$$

and this integral notation is special; the circle through the integral designates an integral over the whole closed surface and the direction of $\hat{n}$ is always taken to be outward. Finally, divide by Δt and take the limit as Δt approaches zero.

$$\frac{dV}{dt} = \oint dA \, \hat{n} \cdot \vec{v} \tag{9.8}$$

The $\vec{v} \cdot \hat{n} \, dA$ is the rate at which the area dA sweeps out volume as it is carried with the fluid. Note: There's nothing in this calculation saying that I have to take the limit as $V \to 0$; it's a perfectly

general expression for the rate of change of volume in a surface being carried with the fluid. It's also a completely general expression for the rate of flow of fluid through a fixed surface as the fluid moves past it. I'm interested in the first interpretation for now, but the second is just as valid in other contexts.

Again, use the standard notation in which the area vector combines the unit normal and the area: $d\vec{A} = \hat{n}\,dA$.
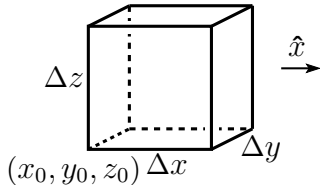
$$\text{divergence of } \vec{v} = \lim_{V \to 0} \frac{1}{V}\frac{dV}{dt} = \lim_{V \to 0} \frac{1}{V} \oint \vec{v} \cdot d\vec{A} \tag{9.9}$$

If the fluid is on average moving away from a point then the divergence there is positive. It's diverging.

The Divergence as Derivatives

This is still a long way from something that you can easily compute. I'll first go through a detailed analysis of how you turn this into a simple result, and I'll then go back to try to capture the essence of the derivation so you can see how it applies in a wide variety of coordinate systems. At that point I'll also show how to get to the result with a lot less algebra. You will see that a lot of the terms that appear in this first calculation will vanish in the end. It's important then to go back and see what was really essential to the calculation and what was not. As you go through this derivation then, try to anticipate which terms are going to be important and which terms are going to disappear.

Express the velocity in rectangular components, $v_x\,\hat{x} + v_y\,\hat{y} + v_z\,\hat{z}$. For the small volume, choose a rectangular box with sides parallel to the axes. One corner is at point (x_0, y_0, z_0) and the opposite corner has coordinates that differ from these by $(\Delta x, \Delta y, \Delta z)$. Expand everything in a power series about the first corner as in section 2.5. Instead of writing out (x_0, y_0, z_0) every time, I'll abbreviate it by $(_0)$.

$$v_x(x, y, z) = v_x(_0) + (x - x_0)\frac{\partial v_x}{\partial x}(_0) + (y - y_0)\frac{\partial v_x}{\partial y}(_0)$$
$$+ (z - z_0)\frac{\partial v_x}{\partial z}(_0) + \frac{1}{2}(x - x_0)^2\frac{\partial^2 v_x}{\partial x^2}(_0) \tag{9.10}$$
$$+ (x - x_0)(y - y_0)\frac{\partial^2 v_x}{\partial x \partial y}(_0) + \cdots$$

There are six integrals to do, one for each face of the box, and there are three functions, v_x, v_y, and v_z to expand in three variables x, y, and z. *Don't Panic.* A lot of these are zero. If you look at the face on the right in the sketch you see that it is parallel to the y-z plane and has normal $\hat{n} = \hat{x}$. When you evaluate $\vec{v} \cdot \hat{n}$, only the v_x term survives; flow parallel to the surface (v_y, v_z) contributes nothing to volume change along this part of the surface. Already then, many terms have simply gone away.

Write the two integrals over the two surfaces parallel to the y-z plane, one at x_0 and one at $x_0 + \Delta x$.

$$\int_{\text{right}} \vec{v} \cdot d\vec{A} + \int_{\text{left}} \vec{v} \cdot d\vec{A}$$
$$= \int_{y_0}^{y_0+\Delta y} dy \int_{z_0}^{z_0+\Delta z} dz\, v_x(x_0 + \Delta x, y, z) - \int_{y_0}^{y_0+\Delta y} dy \int_{z_0}^{z_0+\Delta z} dz\, v_x(x_0, y, z)$$

The minus sign comes from the dot product because $\hat{n}$ points left on the left side. You can evaluate these integrals by using their power series representations, and though you may have an infinite number

of terms to integrate, at least they're all easy. Take the first of them:

$$\int_{y_0}^{y_0+\Delta y} dy \int_{z_0}^{z_0+\Delta z} dz \left[v_x(_0) + \right.$$

$$\left. (\Delta x)\frac{\partial v_x}{\partial x}(_0) + (y-y_0)\frac{\partial v_x}{\partial y}(_0) + (z-z_0)\frac{\partial v_x}{\partial z}(_0) + \frac{1}{2}(\Delta x)^2\frac{\partial^2 v_x}{\partial x^2}(_0) + \cdots \right]$$

$$= v_x(_0)\Delta y\Delta z + (\Delta x)\frac{\partial v_x}{\partial x}(_0)\Delta y\Delta z +$$

$$\frac{1}{2}(\Delta y)^2\Delta z\frac{\partial v_x}{\partial y}(_0) + \frac{1}{2}(\Delta z)^2\Delta y\frac{\partial v_x}{\partial z}(_0) + \frac{1}{2}(\Delta x)^2\frac{\partial^2 v_x}{\partial x^2}(_0)\Delta y\Delta z + \cdots$$

Now look at the second integral, the one that you have to subtract from this one. Before plunging in to the calculation, stop and look around. What will cancel; what will contribute; what will not contribute? The only difference is that this is now evaluated at x_0 instead of at $x_0 + \Delta x$. The terms that have Δx in them simply won't appear this time. All the rest are exactly the same as before. That means that all the terms in the above expression that do *not* have a Δx in them will be canceled when you subtract the second integral. All the terms that *do* have a Δx will be untouched. The combination of the two integrals is then

$$(\Delta x)\frac{\partial v_x}{\partial x}(_0)\Delta y\Delta z + \frac{1}{2}(\Delta x)^2\frac{\partial^2 v_x}{\partial x^2}(_0)\Delta y\Delta z + \frac{1}{2}(\Delta x)\frac{\partial^2 v_x}{\partial x\partial y}(_0)(\Delta y)^2\Delta z + \cdots$$

Two down four to go, but not really. The other integrals are the same except that x becomes y and y becomes z and z becomes x. The integral over the two faces with y constant are then

$$(\Delta y)\frac{\partial v_y}{\partial y}(_0)\Delta z\Delta x + \frac{1}{2}(\Delta y)^2\frac{\partial^2 v_y}{\partial y^2}(_0)\Delta z\Delta x + \cdots$$

and a similar expression for the final two faces. The definition of Eq. (9.9) says to add all three of these expressions, divide by the volume, and take the limit as the volume goes to zero. $V = \Delta x\Delta y\Delta z$, and you see that this is a common factor in all of the terms above. Cancel what you can and you have

$$\frac{\partial v_x}{\partial x}(_0) + \frac{\partial v_y}{\partial y}(_0) + \frac{\partial v_x}{\partial x}(_0) + \frac{1}{2}(\Delta x)\frac{\partial^2 v_x}{\partial x^2}(_0) + \frac{1}{2}(\Delta y)\frac{\partial^2 v_y}{\partial y^2}(_0) + \frac{1}{2}(\Delta z)\frac{\partial^2 v_z}{\partial z^2}(_0) + \cdots$$

In the limit that the all the Δx, Δy, and Δz shrink to zero the terms with a second derivative vanish, as do all the other higher order terms. You are left then with a rather simple expression for the divergence.

$$\text{divergence of } \vec{v} = \text{div}\,\vec{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_x}{\partial x} \tag{9.11}$$

This is abbreviated by using the differential operator ∇, "del."

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \tag{9.12}$$

Then you can write the preceding equation as

$$\text{divergence of } \vec{v} = \text{div}\,\vec{v} = \nabla \cdot \vec{v} \tag{9.13}$$

The symbol ∇ will take other forms in other coordinate systems.

Now that you've waded through this rather technical set of manipulations, is there an easier way? Yes *but*, without having gone through the preceding algebra you won't be able to see and to understand which terms are important and which terms are going to cancel or otherwise disappear. When you need to apply these ideas to something besides rectangular coordinates you have to know what to keep and what to ignore. Once you know this, you can go straight to the end of the calculation and write down those terms that you know are going to survive, dropping the others. *This takes practice.*

Simplifying the derivation

In the long derivation of the divergence, the essence is that you find $\vec{v} \cdot \hat{n}$ on one side of the box (maybe take it in the center of the face), and multiply it by the area of that side. Do this on the other side, remembering that $\hat{n}$ isn't in the same direction there, and combine the results. Do this for each side and divide by the volume of the box.

$$\begin{aligned} \big[v_x(x_0 + \Delta x, & y_0 + \Delta y/2, z_0 + \Delta z/2)\Delta y \Delta z \\ & - v_x(x_0, y_0 + \Delta y/2, z_0 + \Delta z/2)\Delta y \Delta z \big] \div \big(\Delta x \Delta y \Delta z \big) \end{aligned} \tag{9.14}$$

the Δy and Δz factors cancel, and what's left is, in the limit $\Delta x \to 0$, the derivative $\partial v_x / \partial x$.

I was careful to evaluate the values of v_x in the center of the sides, but you see that it didn't matter. In the limit as all the sides go to zero I could just as easily taken the coordinates at one corner and simplified the steps still more. Do this for the other sides, add, and you get the result. It all looks very simple when you do it this way, but what if you need to do it in cylindrical coordinates?
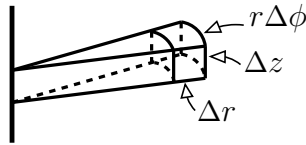
When everything is small, the volume is close to a rectangular box, so its volume is $V = (\Delta r)(\Delta z)(r \Delta \phi)$. Go through the simple version for the calculation of the surface integral. The top and bottom present nothing significantly different from the rectangular case.

$$\big[v_z(r_0, \phi_0, z_0 + \Delta z) - v_z(r_0, \phi_0, z_0) \big](\Delta r)(r_0 \Delta \phi) \div r_0 \Delta r_0 \Delta \phi \Delta z \longrightarrow \frac{\partial v_z}{\partial z}$$

The curved faces of constant r are a bit different, because the areas of the two opposing faces aren't the same.

$$\big[v_r(r_0 + \Delta r, \phi_0, z_0)(r_0 + \Delta r)\Delta \phi \Delta z - v_r(r_0, \phi_0, z_0)r_0 \Delta \phi \Delta z \big] \div r_0 \Delta r \Delta \phi \Delta z \longrightarrow \frac{1}{r}\frac{\partial(r v_r)}{\partial r}$$

A bit more complex than the rectangular case, but not too bad.

Now for the constant ϕ sides. Here the areas of the two faces are the same, so even though they are not precisely parallel to each other this doesn't cause any difficulties.

$$\big[v_\phi(r_0, \phi_0 + \Delta \phi, z_0) - v_z(r_0, \phi_0, z_0) \big](\Delta r)(\Delta z) \div r_0 \Delta r \Delta \phi \Delta z \longrightarrow \frac{1}{r}\frac{\partial v_\phi}{\partial \phi}$$

The sum of all these terms is the divergence expressed in cylindrical coordinates.

$$\text{div } \vec{v} = \frac{1}{r}\frac{\partial(r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} \tag{9.15}$$

The corresponding expression in spherical coordinates is found in exactly the same way, problem 9.4.

$$\text{div } \vec{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{9.16}$$

These are the three commonly occurring coordinates system, though the same simplified method will work in any other orthogonal coordinate system. The coordinate system is orthogonal if the surfaces made by setting the value of the respective coordinates to a constant intersect at right angles. In the spherical example this means that a surface of constant r is a sphere. A surface of constant θ is a half-plane starting from the z-axis. These intersect perpendicular to each other. If you set the third coordinate, ϕ, to a constant you have a cone that intersects the other two at right angles. Look back to section 8.8.

9.4 Integral Representation of Curl
The calculation of the divergence was facilitated by the fact that the equation (9.5) could be manipulated into the form of an integral, Eq. (9.9). Is there a similar expression for the curl? Yes.

$$\text{curl } \vec{v} = \lim_{V\to 0}\frac{1}{V}\oint d\vec{A}\times\vec{v} \tag{9.17}$$

For the divergence there was a logical and orderly development to derive Eq. (9.9) from (9.5). Is there a similar intuitively clear path here? I don't know of one. The best that I can do is to show that it gives the right answer.

And what's that surface integral doing with a $\times$ instead of a $\cdot$? No mistake. Just replace the dot product by a cross product in the definition of the integral. This time however you have to watch the order of the factors.
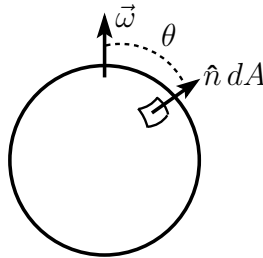
To verify that this does give the correct answer, use a vector field that represents pure rigid body rotation. You're going to take the limit as $\Delta V \to 0$, so it may as well be uniform. The velocity field for this is the same as from problem 7.5.

$$\vec{v} = \vec{\omega}\times\vec{r} \tag{9.18}$$

To evaluate the integral use a sphere of radius R centered at the origin, making $\hat{n} = \hat{r}$. You also need the identity $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$.

$$d\vec{A}\times(\vec{\omega}\times\vec{r}) = \vec{\omega}(d\vec{A}\cdot\vec{r}) - \vec{r}(\vec{\omega}\cdot d\vec{A}) \tag{9.19}$$

Choose a spherical coordinate system with the z-axis along $\vec{\omega}$.

$$d\vec{A} = \hat{n}\,dA = \hat{r}\,dA, \qquad \text{and} \qquad \vec{\omega}\cdot d\vec{A} = \omega\,dA\cos\theta$$

$$\oint d\vec{A} \times \vec{v} = \oint \vec{\omega} R \, dA - \vec{r}\omega \, dA \cos\theta$$

$$= \vec{\omega} R \, 4\pi R^2 - \omega \int_0^\pi R^2 \sin\theta \, d\theta \int_0^{2\pi} d\phi \, \hat{z} R \cos\theta \cos\theta$$

$$= \vec{\omega} R \, 4\pi R^2 - \omega \hat{z} \, 2\pi R^3 \int_0^\pi \sin\theta d\theta \cos^2\theta$$

$$= \vec{\omega} R \, 4\pi R^2 - \omega \hat{z} \, 2\pi R^3 \int_{-1}^1 \cos^2\theta \, d\cos\theta = \vec{\omega} R \, 4\pi R^2 - \omega \hat{z} \, 2\pi R^3 \cdot \frac{2}{3}$$

$$= \vec{\omega}\frac{8}{3}\pi R^3$$

Divide by the volume of the sphere and you have $2\vec{\omega}$ as promised. In the first term on the first line of the calculation, $\vec{\omega} R$ is a constant over the surface so you can pull it out of the integral. In the second term, $\vec{r}$ has components in the $\hat{x}$, $\hat{y}$, and $\hat{z}$ directions; the first two of these integrate to zero because for every vector with a positive $\hat{x}$-component there is one that has a negative component. Same for $\hat{y}$. All that is left of $\vec{r}$ is $\hat{z} R \cos\theta$.

The Curl in Components
With the integral representation, Eq. (9.17), available for the curl, the process is much like that for computing the divergence. Start with rectangular of course. Use the same equation, Eq. (9.10) and the same picture that accompanied that equation. With the experience gained from computing the divergence however, you don't have to go through all the complications of the first calculation. Use the simpler form that followed.

In Eq. (9.14) you have $\vec{v}\cdot\Delta A = v_x\Delta y\Delta z$ on the right face and on the left face. This time replace the dot with a cross (in the right order).

On the right,

$$\Delta\vec{A} \times \vec{v} = \Delta y\Delta z \, \hat{x} \times \vec{v}(x_0 + \Delta x, y_0 + \Delta y/2, z_0 + \Delta z/2) \tag{9.20}$$

On the left it is
$$\Delta\vec{A} \times \vec{v} = \Delta y\Delta z \, \hat{x} \times \vec{v}(x_0, y_0 + \Delta y/2, z_0 + \Delta z/2) \tag{9.21}$$

When you subtract the second from the first and divide by the volume, $\Delta x\Delta y\Delta z$, what is left is (in the limit $\Delta x \to 0$) a derivative.

$$\hat{x} \times \frac{\vec{v}(x_0 + \Delta x, y_0, z_0) - \vec{v}(x_0, y_0, z_0)}{\Delta x} \longrightarrow \hat{x} \times \frac{\partial\vec{v}}{\partial x}$$

$$= \hat{x} \times \left(\hat{x}\frac{\partial v_x}{\partial x} + \hat{y}\frac{\partial v_y}{\partial x} + \hat{z}\frac{\partial v_z}{\partial x} \right)$$

$$= \hat{z}\frac{\partial v_y}{\partial x} - \hat{y}\frac{\partial v_z}{\partial x}$$

Similar calculations for the other four faces of the box give results that you can get simply by changing the labels: $x \to y \to z \to x$, a cyclic permutation of the indices. The result can be expressed most succinctly in terms of ∇.
$$\operatorname{curl} v = \nabla \times \vec{v} \tag{9.22}$$

In the process of this calculation the normal vector $\hat{x}$ was parallel on the opposite faces of the box (except for a reversal of direction). Watch out in other coordinate systems and you'll see that this isn't always true. Just draw the picture in cylindrical coordinates and this will be clear.

9.5 The Gradient

The gradient is the closest thing to an ordinary derivative here, taking a scalar-valued function into a vector field. The simplest geometric definition is "the derivative of a function with respect to distance along the direction in which the function changes most rapidly," and the direction of the gradient vector is along that most-rapidly-changing direction. If you're dealing with one dimension, ordinary real-valued functions of real variables, the gradient *is* the ordinary derivative. Section 8.5 has some discussion and examples of this, including its use in various coordinate systems. It is most conveniently expressed in terms of ∇.

$$\operatorname{grad} f = \nabla f \tag{9.23}$$

The equations (8.15), (8.27), and (8.28) show the gradient (and correspondingly ∇) in three coordinate systems.

$$\text{rectangular:}\quad \nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}$$

$$\text{cylindrical:}\quad \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \tag{9.24}$$

$$\text{spherical:}\quad \nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}$$

In all nine of these components, the denominator (*e.g.* $r\sin\theta\,d\phi$) is the element of displacement along the direction indicated.

9.6 Shorter Cut for div and curl

There is another way to compute the divergence and curl in cylindrical and rectangular coordinates. A direct application of Eqs. (9.13), (9.22), and (9.24) gets the result quickly. The major caution is that you have to be careful that the unit vectors are *inside* the derivative, so you have to differentiate them too.

$\nabla \cdot \vec{v}$ is the divergence of $\vec{v}$, and in cylindrical coordinates

$$\nabla \cdot \vec{v} = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi} + \hat{z}\frac{\partial}{\partial z} \right) \cdot \left(\hat{r}\,v_r + \hat{\phi}\,v_\phi + \hat{z}\,v_z \right) \tag{9.25}$$

The unit vectors $\hat{r}$, $\hat{\phi}$, and $\hat{z}$ don't change as you alter r or z. They do change as you alter ϕ. (except for $\hat{z}$).

$$\frac{\partial \hat{r}}{\partial r} = \frac{\partial \hat{\phi}}{\partial r} = \frac{\partial \hat{z}}{\partial r} = \frac{\partial \hat{r}}{\partial z} = \frac{\partial \hat{\phi}}{\partial z} = \frac{\partial \hat{z}}{\partial z} = \frac{\partial \hat{z}}{\partial \phi} = 0 \tag{9.26}$$

Next come $\partial\hat{r}/\partial\phi$ and $\partial\hat{\phi}/\partial\phi$. This is problem 8.20. You can do this by first showing that

$$\hat{r} = \hat{x}\cos\phi + \hat{y}\sin\phi \qquad \text{and} \qquad \hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi \tag{9.27}$$

and differentiating with respect to ϕ. This gives

$$\partial\hat{r}/\partial\phi = \hat{\phi}, \qquad \text{and} \qquad \partial\hat{\phi}/\partial\phi = -\hat{r} \tag{9.28}$$

Put these together and you have

$$\begin{aligned}
\nabla \cdot \vec{v} &= \frac{\partial v_r}{\partial r} + \hat{\phi} \cdot \frac{1}{r}\frac{\partial}{\partial \phi}\left(\hat{r}\,v_r + \hat{\phi}\,v_\phi \right) + \frac{\partial v_z}{\partial z} \\
&= \frac{\partial v_r}{\partial r} + \hat{\phi} \cdot \frac{1}{r}\left(v_r\frac{d\hat{r}}{d\phi} + \hat{\phi}\frac{\partial v_\phi}{\partial \phi} \right) + \frac{\partial v_z}{\partial z} \\
&= \frac{\partial v_r}{\partial r} + \frac{1}{r}v_r + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z}
\end{aligned} \tag{9.29}$$

This agrees with equation (9.15).

Similarly you can use the results of problem 8.15 to find the derivatives of the corresponding vectors in spherical coordinates. The non-zero values are

$$\frac{d\hat{r}}{d\phi} = \hat{\phi}\sin\theta \qquad \frac{d\hat{\theta}}{d\phi} = \hat{\phi}\cos\theta \qquad \frac{d\hat{\phi}}{d\phi} = -\hat{r}\sin\theta - \hat{\theta}\cos\theta$$

$$\frac{d\hat{r}}{d\theta} = \hat{\theta} \qquad \frac{d\hat{\theta}}{d\theta} = -\hat{r} \tag{9.30}$$

The result is for spherical coordinates

$$\nabla \cdot \vec{v} = \frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta v_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi} \tag{9.31}$$

The expressions for the curl are, cylindrical:

$$\nabla \times \vec{v} = \hat{r}\left(\frac{1}{r}\frac{\partial v_z}{\partial\phi} - \frac{\partial v_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) + \hat{z}\left(\frac{1}{r}\frac{\partial(rv_\phi)}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial\phi}\right) \tag{9.32}$$

and spherical:

$$\nabla \times \vec{v} = \hat{r}\frac{1}{r\sin\theta}\left(\frac{\partial(\sin\theta v_\phi)}{\partial\theta} - \frac{\partial v_\theta}{\partial\phi}\right)$$

$$+\hat{\theta}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{1}{r}\frac{\partial(rv_\phi)}{\partial r}\right) + \hat{\phi}\frac{1}{r}\left(\frac{\partial(rv_\theta)}{\partial r} - \frac{\partial v_r}{\partial\theta}\right) \tag{9.33}$$

9.7 Identities for Vector Operators

Some of the common identities can be proved simply by computing them in rectangular components. These are vectors, and if you show that one vector equals another vector it doesn't matter that you used a simple coordinate system to demonstrate the fact. Of course there are some people who quite properly complain about the inelegance of such a procedure. They're called mathematicians.

$$\nabla \cdot \nabla \times \vec{v} = 0 \qquad \nabla \times \nabla f = 0 \qquad \nabla \times \nabla \times \vec{v} = \nabla(\nabla \cdot \vec{v}) - (\nabla \cdot \nabla)\vec{v} \tag{9.34}$$

There are many other identities, but these are the big three.

$$\oint \vec{v} \cdot d\vec{A} = \int d^3r\, \nabla \cdot \vec{v} \qquad \oint \vec{v} \cdot d\vec{r} = \int \nabla \times \vec{v} \cdot d\vec{A} \tag{9.35}$$

are the two fundamental integral relationships, going under the names of Gauss and Stokes. See chapter 13 for the proofs of these integral relations.

9.8 Applications to Gravity

The basic equations to describe the gravitational field in Newton's theory are

$$\nabla \cdot \vec{g} = -4\pi G\rho, \qquad \text{and} \qquad \nabla \times \vec{g} = 0 \tag{9.36}$$

In these equations, the vector field $\vec{g}$ is defined by placing a (very small) test mass m at a point and measuring the gravitational force on it. This force is proportional to m itself, and the proportionality factor is called the gravitational field $\vec{g}$. The other symbol used here is ρ, and that is the volume mass

density, dm/dV of the matter that is generating the gravitational field. G is Newton's gravitational constant: $G = 6.67 \times 10^{-11}$ N·m^2/kg^2.
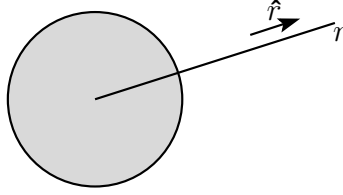
For the first example of solutions to these equations, take the case of a spherically symmetric mass that is the source of a gravitational field. Assume also that its density is constant inside; the total mass is M and it occupies a sphere of radius R. Whatever $\vec{g}$ is, it has only a radial component, $\vec{g} = g_r \hat{r}$. Proof: Suppose it has a sideways component at some point. Rotate the whole system by $180°$ about an axis that passes through this point and through the center of the sphere. The system doesn't change because of this, but the sideways component of $\vec{g}$ would reverse. That can't happen.

The component g_r can't depend on either θ or ϕ because the source doesn't change if you rotate it about any axis; it's spherically symmetric.

$$\vec{g} = g_r(r)\hat{r} \tag{9.37}$$

Now compute the divergence and the curl of this field. Use Eqs. (9.16) and (9.33) to get

$$\nabla \cdot g_r(r)\hat{r} = \frac{1}{r^2}\frac{d(r^2 g_r)}{dr} \qquad \text{and} \qquad \nabla \times g_r(r)\hat{r} = 0$$

The first equation says that the divergence of $\vec{g}$ is proportional to ρ.

$$\frac{1}{r^2}\frac{d(r^2 g_r)}{dr} = -4\pi G\rho \tag{9.38}$$

Outside the surface $r = R$, the mass density is zero, so this is

$$\frac{1}{r^2}\frac{d(r^2 g_r)}{dr} = 0, \qquad \text{implying} \qquad r^2 g_r = C, \qquad \text{and} \qquad g_r = \frac{C}{r^2}$$

where C is some as yet undetermined constant. Now do this inside.

$$\frac{1}{r^2}\frac{d(r^2 g_r)}{dr} = -4\pi G\rho_0, \qquad \text{where} \qquad \rho_0 = 3M/4\pi R^3$$

This is

$$\frac{d(r^2 g_r)}{dr} = -4\pi G\rho_0 \, r^2, \qquad \text{so} \qquad r^2 g_r = -\frac{4}{3}\pi G\rho_0 \, r^3 + C',$$

$$\text{or} \qquad g_r(r) = -\frac{4}{3}\pi G\rho_0 \, r + \frac{C'}{r^2}$$
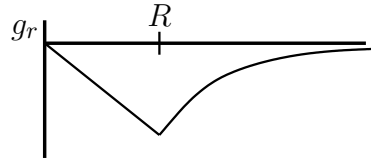
There are two constants that you have to evaluate: C and C'. The latter has to be zero, because $C'/r^2 \to \infty$ as $r \to 0$, and there's nothing in the mass distribution that will cause this. As for the other, note that g_r must be continuous at the surface of the mass. If it isn't, then when you try to differentiate it in Eq. (9.38) you'll be differentiating a step function and you get an infinite derivative there (and the mass density isn't infinite there).

$$g_r(R-) = -\frac{4}{3}\pi G\rho_0 \, R = g_r(R+) = \frac{C}{R^2}$$

Solve for C and you have

$$C = -\frac{4}{3}\pi G\rho_0\, R^3 = -\frac{4}{3}\pi G\frac{3M}{4\pi R^3}R^3 = -GM$$

Put this all together and express the density ρ_0 in terms of M and R to get

$$g_r(r) = \begin{cases} -GM/r^2 & (r > R) \\ -GMr/R^3 & (r < R) \end{cases} \tag{9.39}$$

      This says that outside the spherical mass distribution you can't tell what its radius R is. It creates the same gravitational field as a point mass. Inside the uniform sphere, the field drops to zero linearly toward the center. For a slight variation on how to do this calculation see problem 9.14.

Non-uniform density
The density of the Earth is not uniform; it's bigger in the center. The gravitational field even increases as you go down below the Earth's surface. What does this tell you about the density function? $\nabla \cdot \vec{g} = -4\pi G\rho$ remains true, and I'll continue to make the approximation of spherical symmetry, so this is

$$\frac{1}{r^2}\frac{d\left(r^2 g_r\right)}{dr} = \frac{dg_r}{dr} + \frac{2}{r}g_r = -4\pi G\rho(r) \tag{9.40}$$

That gravity increases with depth (for a little while) says

$$\frac{dg_r}{dr} = -4\pi G\rho(r) - \frac{2}{r}g_r > 0$$

Why > 0? Remember: g_r is itself negative and r is measured outward. Sort out the signs. I can solve for the density to get

$$\rho(r) < -\frac{1}{2\pi Gr}g_r$$

At the surface, $g_r(R) = -GM/R^2$, so this is

$$\rho(R) < \frac{M}{2\pi R^3} = \frac{2}{3}\cdot\frac{3M}{4\pi R^3} = \frac{2}{3}\rho_{\text{average}}$$

The mean density of the Earth is $5.5\,\text{gram}/\text{cm}^3$, so this bound is $3.7\,\text{gram}/\text{cm}^3$. Pick up a random rock. What is its density?

9.9 Gravitational Potential
The gravitational potential is that function V for which

$$\vec{g} = -\nabla V \tag{9.41}$$

That such a function even *exists* is not instantly obvious, but it is a consequence of the second of the two defining equations (9.36). If you grant that, then you can get an immediate equation for V by substituting it into the first of (9.36).

$$\nabla \cdot \vec{g} = -\nabla \cdot \nabla V = -4\pi G\rho, \quad \text{or} \quad \nabla^2 V = 4\pi G\rho \tag{9.42}$$

This is a scalar equation instead of a vector equation, so it will often be easier to handle. Apply it to the same example as above, the uniform spherical mass.

The Laplacian, ∇^2 is the divergence of the gradient, so to express it in spherical coordinates, combine Eqs. (9.24) and (9.31).

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} \qquad (9.43)$$

Because the mass is spherical it doesn't change no matter how you rotate it so the same thing holds for the solution, $V(r)$. Use this spherical coordinate representation of ∇^2 and for this case the θ and ϕ derivatives vanish.

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 4\pi G\rho(r) \qquad (9.44)$$

I changed from ∂ to d because there's now one independent variable, not several. Just as with Eq. (9.38) I'll divide this into two cases, inside and outside.

$$\text{Outside:} \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 0, \qquad \text{so} \qquad r^2\frac{dV}{dr} = C$$

Continue solving this and you have

$$\frac{dV}{dr} = \frac{C}{r^2} \longrightarrow V(r) = -\frac{C}{r} + D \qquad (r > R) \qquad (9.45)$$

$$\text{Inside:} \quad \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = 4\pi G\rho_0 \qquad \text{so} \qquad r^2\frac{dV}{dr} = 4\pi G\rho_0\frac{r^3}{3} + C'$$

Continue, dividing by r^2 and integrating,

$$V(r) = 4\pi G\rho_0\frac{r^2}{6} - \frac{C'}{r} + D' \qquad (r < R) \qquad (9.46)$$

There are now four arbitrary constants to examine. Start with C'. It's the coefficient of $1/r$ in the domain where $r < R$. That means that it blows up as $r \to 0$, but there's nothing at the origin to cause this. $C' = 0$. Notice that the same argument does *not* eliminate C because (9.45) applies only for $r > R$.

Boundary Conditions
Now for the boundary conditions at $r = R$. There are a couple of ways to determine this. I find the simplest and the most general approach is to recognize that the equations (9.42) and (9.44) must be satisfied *everywhere*. That means not just outside, not just inside, but *at* the surface too. The consequence of this statement is the result*

$$V \text{ is continuous at } r = R \qquad\qquad dV/dr \text{ is continuous at } r = R \qquad (9.47)$$

Where do these continuity conditions come from? Assume for a moment that the first one is false, that V is discontinuous at $r = R$, and look at the proposition graphically. If V changes value in a very small interval the graphs of V, of dV/dr, and of d^2V/dr^2 look like

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* Watch out for the similar looking equations that appear in electrostatics. Only the first of these equations holds there; the second must be modified by a dielectric constant.
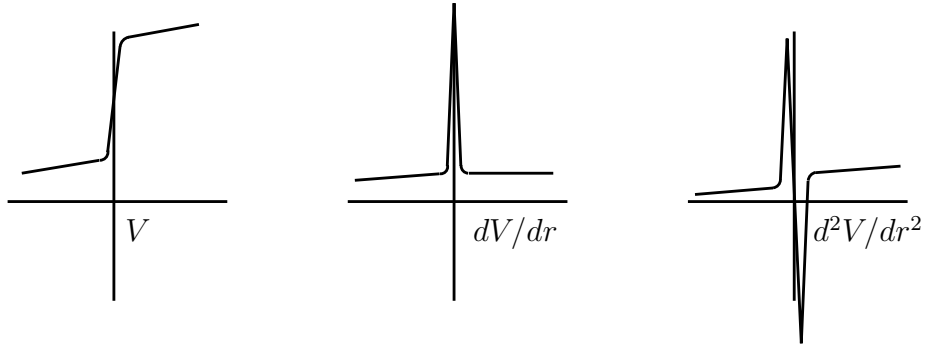
The second derivative on the left side of Eq. (9.44) has a double spike that does not appear on the right side. It can't be there, so my assumption that V is discontinuous is false and V must be continuous.

Assume next that V is continuous but its derivative is not. The graphs of V, of dV/dr, and of d^2V/dr^2 then look like
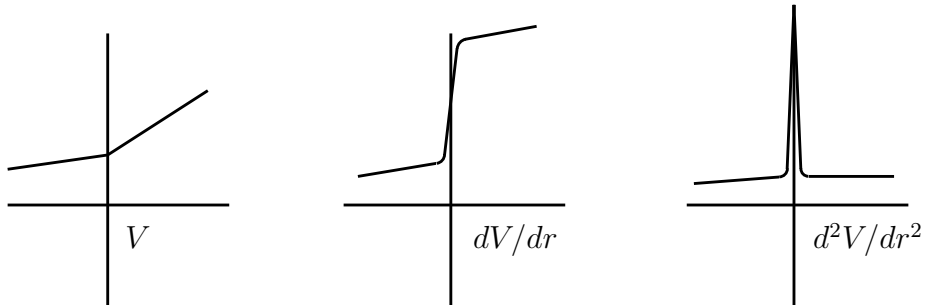
The second derivative on the left side of Eq. (9.44) still has a spike in it and there is no such spike in the ρ on the right side. This is impossible, so dV/dr too must be continuous.

Back to the Problem

Of the four constants that appear in Eqs. (9.45) and (9.46), one is already known, C'. For the rest,

$$V(R-) = V(R+) \quad \text{is} \quad 4\pi G\rho_0 \frac{R^2}{6} + D' = -\frac{C}{R} + D$$

$$\frac{dV}{dr}(R-) = \frac{dV}{dr}(R+) \quad \text{is} \quad 8\pi G\rho_0 \frac{R}{6} = +\frac{C}{R^2}$$

These two equations determine two of the constants.

$$C = 4\pi G\rho_0 \frac{R^3}{3}, \qquad \text{then} \qquad D - D' = 4\pi G\rho_0 \frac{R^2}{6} + 4\pi G\rho_0 \frac{R^2}{3} = 2\pi G\rho_0 R^2$$

Put this together and you have

$$V(r) = \begin{cases} \frac{2}{3}\pi G\rho_0 r^2 - 2\pi G\rho_0 R^2 + D & (r < R) \\ -\frac{4}{3}\pi G\rho_0 R^3/r + D & (r > R) \end{cases} \qquad (9.48)$$

Did I say that the use of potentials is supposed to simplify the problems? Yes, but only the harder problems. The negative gradient of Eq. (9.48) should be $\vec{g}$. Is it? The constant D can't be determined and is arbitrary. You may choose it to be zero.

Magnetic Boundary Conditions

The equations for (time independent) magnetic fields are

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \text{and} \qquad \nabla \cdot \vec{B} = 0 \tag{9.49}$$

The vector $\vec{J}$ is the current density, the current per area, defined so that across a tiny area $d\vec{A}$ the current that flows through the area is $dI = \vec{J} \cdot d\vec{A}$. (This is precisely parallel to Eq. (9.1) for fluid flow rate.) In a wire of radius R, carrying a uniform current I, the magnitude of J is $I/\pi R^2$. These equations are sort of the reverse of Eq. (9.36).

   If there *is* a discontinuity in the current density at a surface such as the edge of a wire, will there be some sort of corresponding discontinuity in the magnetic field? Use the same type of analysis as followed Eq. (9.47) for the possible discontinuities in the potential function. Take the surface of discontinuity of the current density to be the x-y plane, $z = 0$ and write the divergence equation

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$$
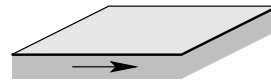
If there is a discontinuity, it will be in the z variable. Perhaps B_x or B_z is discontinuous at the x-y plane. The divergence equation has a derivative with respect to z only on B_z. If one of the other components changes abruptly at the surface, this equation causes no problem — nothing special happens in the x or y direction. If B_z changes at the surface then the derivative $\partial B_z/\partial z$ has a spike. Nothing else in the equation has a spike, so there's no way that you can satisfy the equation. Conclusion: The normal component of $\vec{B}$ is continuous at the surface.

   What does the curl say?

$$\hat{x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial x}\right) + \hat{y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial y}\right) + \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right) = \mu_0\left(\hat{x}\,J_x + \hat{y}\,J_y + \hat{z}\,J_z\right)$$

Derivatives with respect to x or y don't introduce a problem at the surface, all the action is again along z. Only the terms with a $\partial/\partial z$ will raise a question. If B_x is discontinuous at the surface, then its derivative with respect to z will have a spike in the $\hat{y}$ direction that has no other term to balance it. (J_y has a *step* here but not a spike.) Similarly for B_y. This can't happen, so the conclusion: The tangential component of $\vec{B}$ is continuous at the surface.

   What if the surface isn't a plane. Maybe it is a cylinder or a sphere. In a small enough region, both of these look like planes. That's why there is still* a Flat Earth Society.

9.10 Index Notation

In section 7.11 I introduced the summation convention for repeated indices. I'm now going to go over it again and emphasize its utility in practical calculations.

   When you want to work in a rectangular coordinate system, with basis vectors $\hat{x}$, $\hat{y}$, and $\hat{z}$, it is convenient to use a more orderly notation for the basis vectors instead of just a sequence of letters of the alphabet. Instead, call them $\hat{e}_1$, $\hat{e}_2$, and $\hat{e}_3$. (More indices if you have more dimensions.) I'll keep the assumption that these are orthogonal unit vectors so that

$$\hat{e}_1 \cdot \hat{e}_2 = 0, \qquad \hat{e}_3 \cdot \hat{e}_3 = 1, \;\; \textit{etc.}$$

More generally, write this in the compact notation

$$\hat{e}_i \cdot \hat{e}_j = \delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{9.50}$$

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   * en.wikipedia.org/wiki/Flat_Earth_Society

The Kronecker delta is either one or zero depending on whether $i = j$ or $i \neq j$, and this equation sums up the properties of the basis in a single, compact notation. You can now write a vector in this basis as

$$\vec{A} = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3 = A_i \hat{e}_i$$

The last expression uses the summation convention that a repeated index is summed over its range. When an index is repeated in a term, you will invariably have exactly two instances of the index; if you have three it's a mistake.

When you add or subtract vectors, the index notation is

$$\vec{A} + \vec{B} = \vec{C} = A_i \hat{e}_i + B_i \hat{e}_i = (A_i + B_i)\hat{e}_i = C_i \hat{e}_i \qquad \textbf{or} \qquad A_i + B_i = C_i$$

Does it make sense to write $\vec{A} + \vec{B} = A_i \hat{e}_i + B_k \hat{e}_k$? Yes, but it's sort of pointless and confusing. You can change any summed index to any label you find convenient — they're just dummy variables.

$$A_i \hat{e}_i = A_\ell e_\ell = A_m \hat{e}_m = A_1 \hat{e}_1 + A_2 \hat{e}_2 + A_3 \hat{e}_3$$

You can sometime use this freedom to help do manipulations, but in the example $A_i \hat{e}_i + B_k \hat{e}_k$ it's no help at all.

Combinations such as

$$E_i + F_i \qquad \text{or} \qquad E_k F_k G_i = H_i \qquad \text{or} \qquad M_{k\ell} D_\ell = F_k$$

are valid. The last is simply Eq. (7.8) for a matrix times a column matrix.

$$A_i + B_j = C_k \qquad \text{or} \qquad E_m F_m G_m \qquad \text{or} \qquad C_k = A_{ij} B_j$$

have no meaning.

You can manipulate the indices for your convenience as long as you follow the rules.

$$A_i = B_{ij} C_j \qquad \text{is the same as} \qquad A_k = B_{kn} C_n \qquad \text{or} \qquad A_\ell = B_{\ell p} C_p$$

The scalar product has a simple form in index notation:

$$\vec{A} \cdot \vec{B} = A_i \hat{e}_i \cdot B_j \hat{e}_j = A_i B_j \hat{e}_i \cdot \hat{e}_j = A_i B_j \delta_{ij} = A_i B_i \tag{9.51}$$

The final equation comes by doing one of the two sums (say j), and only the term with $j = i$ survives, producing the final expression. The result shows that the sum over the repeated index is the scalar product in disguise.

Just as the dot product of the basis vectors is the delta-symbol, the cross product provides another important index function, the alternating symbol.

$$\hat{e}_i \cdot \hat{e}_j \times \hat{e}_k = \epsilon_{ijk} \tag{9.52}$$

$$\hat{e}_2 \times \hat{e}_3 \equiv \hat{y} \times \hat{z} = \hat{e}_1, \qquad \text{so} \qquad \epsilon_{123} = \hat{e}_1 \cdot \hat{e}_2 \times \hat{e}_3 = 1$$
$$\hat{e}_2 \times \hat{e}_3 = -\hat{e}_3 \times \hat{e}_2, \qquad \text{so} \qquad \epsilon_{132} = -\epsilon_{123} = -1$$
$$\hat{e}_3 \cdot \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3 = 1, \qquad \text{so} \qquad \epsilon_{312} = \epsilon_{123} = 1$$
$$\hat{e}_1 \cdot \hat{e}_3 \times \hat{e}_3 = \hat{e}_1 \cdot 0 = 0 \qquad \text{so} \qquad \epsilon_{133} = 0$$

If the indices are a cyclic permutation of 123, (231 or 312), the alternating symbol is 1.
If the indices are an odd permutation of 123, (132 or 321 or 213), the symbol is -1.
If any two of the indices are equal the alternating symbol is zero, and that finishes all the cases. The last property is easy to see because if you interchange any two indices the sign changes. If the indices are the same, the sign can't change so it must be zero.

Use the alternating symbol to write the cross product itself.

$$\vec{A} \times \vec{B} = A_i \hat{e}_i \times B_j \hat{e}_j \qquad \text{and the } k\text{-component is}$$
$$\hat{e}_k \cdot \vec{A} \times \vec{B} = \hat{e}_k \cdot A_i B_j \hat{e}_i \times \hat{e}_j = \epsilon_{kij} A_i B_j$$

You can use the summation convention to advantage in calculus too. The ∇ vector operator has components

$$\nabla_i \qquad \text{or some people prefer} \qquad \partial_i$$

For unity of notation, use $x_1 = x$ and $x_2 = y$ and $x_3 = z$. In this language,

$$\partial_1 \equiv \nabla_1 \qquad \text{is} \qquad \frac{\partial}{\partial x} \equiv \frac{\partial}{\partial x_1} \tag{9.53}$$

Note: This notation applies to rectangular component calculations only! ($\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$.) The generalization to curved coordinate systems will wait until chapter 12.

$$\text{div } \vec{v} = \nabla \cdot \vec{v} = \partial_i v_i = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \tag{9.54}$$

You should verify that $\partial_i x_j = \delta_{ij}$.

Similarly the curl is expressed using the alternating symbol.

$$\text{curl } \vec{v} = \nabla \times \vec{v} \qquad \text{becomes} \qquad \epsilon_{ijk} \partial_j v_k = \big(\text{curl } \vec{v} \big)_i \tag{9.55}$$

the ith components of the curl.

An example of manipulating these object: Prove that $\text{curl grad } \phi = 0$.

$$\text{curl grad } \phi = \nabla \times \nabla \phi \longrightarrow \epsilon_{ijk} \partial_j \partial_k \phi \tag{9.56}$$

You can interchange the order of the differentiation as always, and the trick here is to relabel the indices — that is a standard technique in this business.

$$\epsilon_{ijk} \partial_j \partial_k \phi = \epsilon_{ijk} \partial_k \partial_j \phi = \epsilon_{ikj} \partial_j \partial_k \phi$$
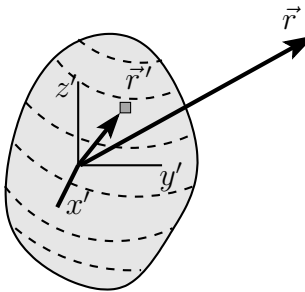
The first equation interchanges the order of differentiation. In the second equation, I call "j" "k" and I call "k" "j". These are dummy indices, summed over, so this can't affect the result, but now this expression looks like that of Eq. (9.56) except that two (dummy) indices are reversed. The ϵ symbol is antisymmetric under interchange of any of its indices, so the final expression is the negative of the expression in Eq. (9.56). The only number equal to minus itself is zero, so the identity is proved.

9.11 More Complicated Potentials
The gravitational field from a point mass is $\vec{g} = -Gm\hat{r}/r^2$, so the potential for this point mass is $\phi = -Gm/r$. This satisfies

$$\vec{g} = -\nabla\phi = -\nabla\frac{-Gm}{r} = \hat{r}\frac{\partial}{\partial r}\frac{Gm}{r} = -\frac{Gm\hat{r}}{r^2}$$

For several point masses, the gravitational field is the vector sum of the contributions from each mass. In the same way the gravitational potential is the (scalar) sum of the potentials contributed by each mass. This is almost always easier to calculate than the vector sum. If the distribution is continuous, you have an integral.

$$\phi_{\text{total}} = \sum -\frac{Gm_k}{r_k} \qquad \text{or} \qquad -\int \frac{G\,dm}{r}$$

This sort of very abbreviated notation for the sums and integrals is normal once you have done a lot of them, but when you're just getting started it is useful to go back and forth between this terse notation and a more verbose form. Expand the notation and you have

$$\phi_{\text{total}}(\vec{r}) = -G\int \frac{dm}{|\vec{r} - \vec{r}'|} \tag{9.57}$$

This is still not very explicit, so expand it some more. Let

$$\vec{r}' = \hat{x}\,x' + \hat{y}\,y' + \hat{z}\,z' \qquad \text{and} \qquad \vec{r} = \hat{x}\,x + \hat{y}\,y + \hat{z}\,z$$

$$\text{then} \qquad \phi(x,y,z) = -G\int dx'dy'dz'\,\rho(x',y',z')\frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

where ρ is the volume mass density so that $dm = \rho\,dV = \rho\,d^3r'$, and the limits of integration are such that this extends over the whole volume of the mass that is the source of the potential. The primed coordinates represent the positions of the masses, and the non-primed ones are the position of the point where you are evaluating the potential, the field point. The combination d^3r is a common notation for a volume element in three dimensions.

For a simple example, what is the gravitational potential from a uniform thin rod? Place its center at the origin and its length $= 2L$ along the z-axis. The potential is

$$\phi(\vec{r}) = -\int \frac{G\,dm}{r} = -G\int \frac{\lambda dz'}{\sqrt{x^2 + y^2 + (z - z')^2}}$$

where $\lambda = M/2L$ is its linear mass density. This is an elementary integral. Let $u = z' - z$, and $a = \sqrt{x^2 + y^2}$.

$$\phi = -G\lambda\int_{-L-z}^{L-z} \frac{du}{\sqrt{a^2 + u^2}} = -G\lambda\int d\theta = -G\lambda\,\theta\,\Big|_{u=-L-z}^{u=L-z}$$

where $u = a\sinh\theta$. Put this back into the original variables and you have

$$\phi = -G\lambda\left[\sinh^{-1}\left(\frac{L-z}{\sqrt{x^2+y^2}}\right) + \sinh^{-1}\left(\frac{L+z}{\sqrt{x^2+y^2}}\right)\right] \tag{9.58}$$

The inverse hyperbolic function is a logarithm as in Eq. (1.4), so this can be rearranged and the terms combined into the logarithm of a function of x, y, and z, but the $\sinh^{-1}$s are easier to work with so there's not much point. This is not too complicated a result, and it is far easier to handle than the vector field you get if you take its gradient. It's still necessary to analyze it in order to understand it and to check for errors. See problem 9.48.

Exercises

1 Prove that the geometric interpretation of a cross product is as an area.

2 Start from a picture of $\vec{C} = \vec{A} - \vec{B}$ and use the definition and properties of the dot product to derive the law of cosines. (If this takes you more than about three lines, start over, and no components, just vectors.)

3 Start from a picture of $\vec{C} = \vec{A} - \vec{B}$ and use the definition and properties of the cross product to derive the law of sines. (If this takes you more than a few lines, start over.)

4 Show that $\vec{A}\cdot\vec{B}\times\vec{C} = \vec{A}\times\vec{B}\cdot\vec{C}$. Do this by drawing pictures of the three vectors and find the geometric meaning of each side of the equation, showing that they are the same (including sign).

5 (a) If the dot product of a given vector $\vec{F}$ with every vector results in zero, show that $\vec{F} = 0$. (b) Same for the cross product.

6 From the definition of the dot product, and in two dimensions, draw pictures to interpret $\vec{A}\cdot(\vec{B}+\vec{C})$ and from there prove the distributive law: $\vec{A}\cdot = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$. (Draw $\vec{B}+\vec{C}$ tip-to-tail.)

7 For a sphere, from the definition of the integral, what is $\oint d\vec{A}$? What is $\oint dA$?

8 What is the divergence of $\hat{x}\,xy + \hat{y}\,yz + \hat{z}\,zx$?

9 What is the divergence of $\hat{r}\,r\sin\theta + \hat{\theta}\,r_0\sin\theta\cos\phi + \hat{\phi}\,r\cos\phi$? (spherical)

10 What is the divergence of $\hat{r}\,r\sin\phi + \hat{\phi}\,z\sin\phi + \hat{z}\,zr$? (cylindrical)

11 In cylindrical coordinates draw a picture of the vector field $\vec{v} = \hat{\phi}r^2(2+\cos\phi)$ (for $z=0$). Compute the divergence of $\vec{v}$ and indicate in a second sketch what it looks like. Draw field lines for $\vec{v}$.

12 What is the curl of the vector field in the preceding exercise (and indicate in a sketch what it's like).

13 Calculate $\nabla\cdot\hat{r}$. (a) in cylindrical and (b) in spherical coordinates.

14 Compute $\partial_i x_j$.

15 Compute div curl $\vec{v}$ using index notation.

16 Show that $\epsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i)$.

17 Use index notation to derive $\nabla\cdot(f\vec{v}) = (\nabla f)\cdot\vec{v} + f\nabla\cdot\vec{v}$.

18 Write Eq. (8.6) in index notation, where $(x,y)\to x_i$ and $(x',y')\to x'_j$.

Problems

9.1 Use the same geometry as that following Eq. (9.3), and take the velocity function to be $\vec{v} = \hat{x}\,v_0 xy/b^2$. Take the bottom edge of the plane to be at $(x, y) = (0, 0)$ and calculate the flow rate. Should the result be independent of the angle ϕ? Sketch the flow to understand this point. Does the result check for any special, simple value of ϕ? Ans: $(v_0 ab \tan \phi)/3$

9.2 Repeat the preceding problem using the cylindrical surface of Eq. (9.4), but place the bottom point of the cylinder at coordinate $(x, y) = (x_0, 0)$. Ans: $(v_0 a/4)(2x_0 + \pi b/4)$

9.3 Use the same velocity function $\vec{v} = \hat{x}\,v_0 xy/b^2$ and evaluate the flow integral outward from the *closed* surface of the rectangular box, $(c < x < d)$, $(0 < y < b)$, $(0 < z < a)$. The convention is that the unit normal vector points outward from the six faces of the box. Ans: $v_0 a(d - c)/2$

9.4 Work out the details, deriving the divergence of a vector field in spherical coordinates, Eq. (9.16).

9.5 (**a**) For the vector field $\vec{v} = A\vec{r}$, that is pointing away from the origin with a magnitude proportional to the distance from the origin, express this in rectangular components and compute its divergence.
(**b**) Repeat this in cylindrical coordinates (still pointing away from the origin though).
(**c**) Repeat this in spherical coordinates, Eq. (9.16).

9.6 Gauss's law for electromagnetism says $\oint \vec{E} \cdot d\vec{A} = q_{encl}/\epsilon_0$. If the electric field is given to be $\vec{E} = A\vec{r}$, what is the surface integral of $\vec{E}$ over the whole closed surface of the cube that spans the region from the origin to $(x, y, z) = (a, a, a)$?
(**a**) What is the charge enclosed in the cube?
(**b**) Compute the volume integral, $\int d^3r\, \nabla \cdot \vec{E}$ inside the same cube?

9.7 Evaluate the surface integral, $\oint \vec{v} \cdot d\vec{A}$, of $\vec{v} = \hat{r}\,Ar^2 \sin^2 \theta + \hat{\theta}\,Br \cos \theta \sin \phi$ over the surface of the sphere centered at the origin and of radius R. Recall section 8.8.

9.8 (**a**) What is the area of the spherical cap on the surface of a sphere of radius R: $0 \leq \theta \leq \theta_0$?
(**b**) Does the result have the correct behavior for both small and large θ_0?
(**c**) What are the surface integrals over this cap of the vector field $\vec{v} = \hat{r}\,v_0 \cos \theta \sin^2 \phi$? Consider both $\int \vec{v} \cdot d\vec{A}$ and $\int \vec{v} \times d\vec{A}$. Ans: $v_0 \pi R^2 (1 - \cos^2 \theta_0)/2$

9.9 A rectangular area is specified parallel to the x-y plane at $z = d$ and $0 < x < a$, $a < y < b$. A vector field is $\vec{v} = (\hat{x}\,Axyz + \hat{y}\,Byx^2 + \hat{z}\,Cx^2 yz^2)$ Evaluate the two integrals over this surface

$$\int \vec{v} \cdot d\vec{A}, \qquad \text{and} \qquad \int d\vec{A} \times \vec{v}$$

9.10 For the vector field $\vec{v} = Ar^n \vec{r}$, compute the integral over the surface of a sphere of radius R centered at the origin: $\oint \vec{v} \cdot d\vec{A}$. $(n \geq 0)$
Compute the integral over the volume of this same sphere $\int d^3r\, \nabla \cdot \vec{v}$.

9.11 The velocity of a point in a rotating rigid body is $\vec{v} = \vec{\omega} \times \vec{r}$. See problem 7.5. Compute its divergence and curl. Do this in rectangular, cylindrical, and spherical coordinates.

9.12 Fill in the missing steps in the calculation of Eq. (9.29).

9.13 Mimic the calculation in section 9.6 for the divergence in cylindrical coordinates, computing the curl in cylindrical coordinates, $\nabla \times \vec{v}$. Ans: Eq. (9.32).

9.14 Another way to get to Eq. (9.39) is to work with Eq. (9.38) directly and to write the function $\rho(r)$ explicitly as two cases: $r < R$ and $r > R$. Multiply Eq. (9.38) by r^2 and integrate it from zero to r, being careful to handle the integral differently when the upper limit is $< R$ and when it is $> R$.

$$r^2 g_r(r) = -4\pi G \int_0^r dr' \, r'^2 \rho(r')$$

Note: This is not simply reproducing that calculation that I've already done. This is doing it a different way.

9.15 If you have a very large (assume it's infinite) slab of mass of thickness d the gravitational field will be perpendicular to its plane. To be specific, say that there is a uniform mass density ρ_0 between $z = \pm d/2$ and that $\vec{g} = g_z(z)\,\hat{z}$. Use Eqs. (9.36) to find $g_z(z)$.
Be precise in your reasoning when you evaluate any constants. (What happens when you rotate the system about the x-axis?) Does your graph of the result make sense?
Ans: in part, $g_z = +2\pi G \rho_0 d$, $(z < -d/2)$

9.16 Use Eqs. (9.36) to find the gravitational field of a very long solid cylinder of uniform mass density ρ_0 and radius R. (Assume it's infinitely long.) Start from the assumption that in cylindrical coordinates the field is $\vec{g} = g_r(r, \phi, z)\,\hat{r}$, and apply both equations.
Ans: in part $g_r = -2\pi G \rho_0 r$, $(0 < r < R)$

9.17 The gravitational field in a spherical region $r < R$ is stated to be $\vec{g}(r) = -\hat{r}\,C/r$, where C is a constant. What mass density does this imply?
If there is no mass for $r > R$, what is $\vec{g}$ there?

9.18 In Eq. (8.23) you have an approximate expression for the gravitational field of Earth, including the effect of the equatorial bulge. Does it satisfy Eqs. (9.36)? $(r > R_{\text{Earth}})$

9.19 Compute the divergence of the velocity function in problem 9.3 and integrate this divergence over the volume of the box specified there. Ans: $(d - c)av_0$

9.20 The gravitational potential, equation (9.42), for the case that the mass density is zero says to set the Laplacian Eq. (9.43) equal to zero. Assume a solution to $\nabla^2 V = 0$ to be a function of the spherical coordinates r and θ alone and that

$$V(r, \theta) = Ar^{-(\ell+1)} f(x), \qquad \text{where} \qquad x = \cos\theta$$

Show that this works provided that f satisfies a certain differential equation and show that it is the Legendre equation of Eq. (4.18) and section 4.11.

9.21 The volume energy density, dU/dV in the electric field is $\epsilon_0 E^2/2$. The electrostatic field equations are the same as the gravitational field equations, Eq. (9.36).

$$\nabla \cdot \vec{E} = \rho/\epsilon_0, \qquad \text{and} \qquad \nabla \times \vec{E} = 0$$

A uniformly charged ball of radius R has charge density ρ_0 for $r < R$, $Q = 4\pi \rho_0 R^3/3$.
(**a**) What is the electric field everywhere due to this charge distribution?

(**b**) The total energy of this electric field is the integral over all space of the energy density. What is it?
(**c**) If you want to account for the mass of the electron by saying that all this energy that you just computed *is* the electron's mass via $E_0 = mc^2$, then what must the electron's radius be? What is its numerical value? Ans: $r_e = \frac{3}{5}\left(e^2/4\pi\epsilon_0 mc^2\right) = 1.69\,\text{fm}$

9.22 The equations relating a magnetic field, $\vec{B}$, to the current producing it are, for the stationary case,

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \text{and} \qquad \nabla \cdot \vec{B} = 0$$

Here $\vec{J}$ is the current density, current per area, defined so that across a tiny area $d\vec{A}$ the current that flows through the area is $dI = \vec{J} \cdot d\vec{A}$. A cylindrical wire of radius R carries a total current I distributed uniformly across the cross section of the wire. Put the z-axis of a cylindrical coordinate system along the central axis of the wire with positive z in the direction of the current flow. Write the function $\vec{J}$ explicitly in these coordinates (for all values of $r < R$, $r > R$). Use the curl and divergence expressed in cylindrical coordinates and assume a solution in the form $\vec{B} = \hat{\phi}\,B_\phi(r, \phi, z)$. Write out the divergence and curl equations and show that you can satisfy these equations relating $\vec{J}$ and $\vec{B}$ with such a form, solving for B_ϕ. Sketch a graph of the result. At a certain point in the calculation you will have to match the boundary conditions at $r = R$. Recall that the tangential component of $\vec{B}$ (here B_ϕ) is continuous at the boundary.
Ans: in part, $\mu_0 Ir/2\pi R^2 \ \ (r < R)$

9.23 A long cylinder of radius R has a uniform charge density inside it, ρ_0 and it is rotating about its long axis with angular speed ω. This provides an azimuthal current density $\vec{J} = \rho_0 r\omega\,\hat{\phi}$ in cylindrical coordinates. Assume the form of the magnetic field that this creates has only a z-component: $\vec{B} = B_z(r, \phi, z)\hat{z}$ and apply the equations of the preceding problem to determine this field both inside and outside. The continuity condition at $r = R$ is that the tangential component of $\vec{B}$ (here it is B_z) is continuous there. The divergence and the curl equations will (almost) determine the rest. Ans: in part, $-\rho r^2 \omega/2 + C \ \ (r < R)$

9.24 By analogy to Eqs. (9.9) and (9.17) the expression

$$\lim_{V \to 0} \frac{1}{V} \oint \phi\, d\vec{A}$$

is the gradient of the scalar function ϕ. Compute this in rectangular coordinates by mimicking the derivation that led to Eq. (9.11) or (9.15), showing that it has the correct components.

9.25 (**a**) A fluid of possibly non-uniform mass density is in equilibrium in a possibly non-uniform gravitational field. Pick a volume and write down the total force vector on the fluid in that volume; the things acting on it are gravity and the surrounding fluid. Take the limit as the volume shrinks to zero, and use the result of the preceding problem in order to get the equation for equilibrium.
(**b**) Now apply the result to the special case of a uniform gravitational field and a constant mass density to find the pressure variation with height. Starting from an atmospheric pressure of $1.01 \times 10^5\,\text{N/m}^2$, how far must you go under water to reach double this pressure?
Ans: $\nabla p = -\rho\vec{g}$; about 10 meters

9.26 The volume energy density, $u = dU/dV$, in the gravitational field is $g^2/8\pi G$. [Check the units to see if it makes sense.] Use the results found in Eq. (9.39) for the gravitational field of a spherical mass and get the energy density. An extension of Newton's theory of gravity is that the source of

gravity is *energy* not just mass! This energy that you just computed from the gravitational field is then the source of more gravity, and this energy density contributes as a mass density $\rho = u/c^2$ would.
(**a**) Find the additional gravitational field $g_r(r)$ that this provides and add it to the previous result for $g_r(r)$.
(**b**) For our sun, its mass is 2×10^{30} kg and its radius is $700,000$ km. Assume its density is constant throughout so that you can apply the results of this problem. At the sun's surface, what is the ratio of this correction to the original value?
(**c**) What radius would the sun have to be so that this correction is equal to the original $g_r(R)$, resulting in double gravity? Ans: (**a**) $g_{\text{correction}}/g_{\text{original}} = GM/10Rc^2$

9.27 Continuing the ideas of the preceding problem, the energy density, $u = dU/dV$, in the gravitational field is $g^2/8\pi G$, and the source of gravity is energy not just mass. In the region of space that is empty of matter, show that the divergence equation for gravity, (9.36), then becomes

$$\nabla \cdot \vec{g} = -4\pi Gu/c^2 = -g^2/2c^2$$

Assume that you have a spherically symmetric system, $\vec{g} = g_r(r)\hat{r}$, and write the differential equation for g_r.
(**a**) Solve it and apply the boundary condition that as $r \to \infty$, the gravitational field should go to $g_r(r) \to -GM/r^2$. How does this solution behave as $r \to 0$ and compare its behavior to that of the usual gravitational field of a point mass.
(**b**) Can you explain why the behavior is different? Note that in this problem it is the gravitational field itself that is the source of the gravitational field; mass as such isn't present.
(**c**) A characteristic length appears in this calculation. Evaluate it for the sun. It is $1/4$ the Schwarzchild radius that appears in the theory of general relativity.
Ans: (**a**) $g_r = -GM/[r(r+R)]$, where $R = GM/2c^2$

9.28 In the preceding problem, what is the total energy in the gravitational field, $\int u\, dV$? How does this ($\div c^2$) compare to the mass M that you used in setting the value of g_r as $r \to \infty$?

9.29 Verify that the solution Eq. (9.48) does satisfy the continuity conditions on V and V'.

9.30 The r-derivatives in Eq. (9.43), spherical coordinates, can be written in a different and more convenient form. Show that they are equivalent to

$$\frac{1}{r}\frac{\partial^2 (rV)}{\partial r^2}$$

9.31 The gravitational potential from a point mass M is $-GM/r$ where r is the distance to the mass. Place a single point mass at coordinates $(x, y, z) = (0, 0, d)$ and write its potential V. Write this expression in terms of spherical coordinates about the origin, (r, θ), and then expand it for the case $r > d$ in a power series in d/r, putting together the like powers of d/r. Do this through order $(d/r)^3$. Express the result in the language of Eq. (4.61).
Ans: $-\frac{GM}{r} - \frac{GMd}{r^2}[\cos\theta] - \frac{GMd^2}{r^3}[\frac{3}{2}\cos^2\theta - \frac{1}{2}] - \frac{GMd^3}{r^4}[\frac{5}{2}\cos^3\theta - \frac{3}{2}\cos\theta]$

9.32 As in the preceding problem a point mass M has potential $-GM/r$ where r is the distance to the mass. The mass is at coordinates $(x, y, z) = (0, 0, d)$. Write its potential V in terms of spherical coordinates about the origin, (r, θ), but this time take $r < d$ and expand it in a power series in r/d. Do this through order $(r/d)^3$.
Ans: $(-GM/d)[1 + (r/d)P_1(\cos\theta) + (r^2/d^2)P_2(\cos\theta) + (r^3/d^3)P_3(\cos\theta) + \cdots]$

9.33 Theorem: Given that a vector field satisfies $\nabla \times \vec{v} = 0$ everywhere, then it follows that you can write $\vec{v}$ as the gradient of a scalar function, $\vec{v} = -\nabla\psi$. For each of the following vector fields find if possible, and probably by trail and error if so, a function ψ that does this. *First* determine is the curl is zero, because if it isn't then your hunt for a ψ will be futile. You should try however — the hunt will be instructive.

(a) $\hat{x}\, y^3 + 3\hat{y}\, xy^2$,   (c) $\hat{x}\, y\cos(xy) + \hat{y}\, x\cos(xy)$,

(b) $\hat{x}\, x^2 y + \hat{y}\, xy^2$,   (d) $\hat{x}\, y^2 \sinh(2xy^2) + 2\hat{y}\, xy\sinh(2xy^2)$

9.34 A hollow sphere has inner radius a, outer radius b, and mass M, with uniform mass density in this region.
(**a**) Find (and sketch) its gravitational field $g_r(r)$ everywhere.
(**b**) What happens in the limit that $a \to b$? In this limiting case, graph g_r. Use $g_r(r) = -dV/dr$ and compute and graph the potential function $V(r)$ for this limiting case. This violates Eq. (9.47). Why?
(**c**) Compute the area mass density, $\sigma = dM/dA$, in this limiting case and find the relationship between the discontinuity in dV/dr and the value of σ.

9.35 Evaluate

$$\delta_{ij}\epsilon_{ijk}, \qquad \epsilon_{mjk}\epsilon_{njk}, \qquad \partial_i x_j, \qquad \partial_i x_i, \qquad \epsilon_{ijk}\epsilon_{ijk}, \qquad \delta_{ij}v_j$$

and show that $\epsilon_{ijk}\epsilon_{mnk} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$

9.36 Verify the identities for arbitrary $\vec{A}$,

$$
\begin{aligned}
(\vec{A}\cdot\nabla)\vec{r} = \vec{A} &\qquad \text{or} \qquad A_i\partial_i x_j = A_j \\
\nabla\cdot\nabla\times\vec{v} = 0 &\qquad \text{or} \qquad \partial_i\epsilon_{ijk}\partial_j v_k = 0 \\
\nabla\cdot(f\vec{A}) = (\nabla f)\cdot\vec{A} + f(\nabla\cdot\vec{A}) &\qquad \text{or} \qquad \partial_i(fA_i) = (\partial_i f)A_i + f\partial_i A_i
\end{aligned}
$$

You can try proving all these in the standard vector notation, but use the index notation instead. It's a lot easier.

9.37 Use index notation to prove $\nabla\times\nabla\times\vec{v} = \nabla(\nabla\cdot\vec{v}) - \nabla^2\vec{v}$. First, what identity you have to prove about ϵ's.

9.38 Is $\nabla\times\vec{v}$ perpendicular to $\vec{v}$? Either prove it's true or give an explicit example for which it is false.

9.39 If for arbitrary A_i and arbitrary B_j it is known that $a_{ij}A_iB_j = 0$, prove then that all the a_{ij} are zero.

9.40 Compute the divergences of
$$Ax\,\hat{x} + By^2\,\hat{y} + C\hat{z} \text{ in rectangular coordinates.}$$
$$Ar\,\hat{r} + B\theta^2\,\hat{\theta} + C\hat{\phi} \text{ in spherical coordinates.}$$
How do the pictures of these vector fields correspond to the results of these calculations?

9.41 Compute the divergence and the curl of

$$\frac{y\,\hat{x} - x\,\hat{y}}{x^2 + y^2}, \qquad \text{and of} \qquad \frac{y\,\hat{x} - x\,\hat{y}}{(x^2 + y^2)^2}$$

9.42 Translate the preceding vector fields into polar coordinates, then take their divergence and curl. And of course draw some pictures.

9.43 As a review of ordinary vector algebra, and perhaps some practice in using index notation, translate the triple scalar product into index notation and prove first that it is invariant under cyclic permutations of the vectors.
(**a**) $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{B} \cdot \vec{C} \times \vec{A} = \vec{C} \cdot \vec{A} \times \vec{B}$. Then that
(**b**) $\vec{A} \cdot \vec{B} \times \vec{C} = \vec{A} \times \vec{B} \cdot \vec{C}$.
(**c**) What is the result of interchanging any pair of the vectors in the product?
(**d**) Show why the geometric interpretation of this product is as the volume of a parallelepiped.

9.44 What is the total flux, $\oint \vec{E} \cdot d\vec{A}$, out of the cube of side a with one corner at the origin?
(**a**) $\vec{E} = \alpha \hat{x} + \beta \hat{y} + \gamma \hat{z}$
(**b**) $\vec{E} = \alpha x \, \hat{x} + \beta y \, \hat{y} + \gamma z \, \hat{z}$.

9.45 The electric potential from a single point charge q is kq/r. Two charges are on the z-axis: $-q$ at position $z = z_0$ and $+q$ at position $z_0 + a$.
(**a**) Write the total potential at the point (r, θ, ϕ) in spherical coordinates.
(**b**) Assume that $r \gg a$ and $r \gg z_0$, and use the binomial expansion to find the series expansion for the total potential out to terms of order $1/r^3$.
(**c**) how does the coefficient of the $1/r^2$ term depend on z_0? The coefficient of the $1/r^3$ term? These tell you the total electric dipole moment and the total quadrupole moment.
(**d**) What is the curl of the gradient of each of these two terms?
The polynomials of section 4.11 will appear here, with argument $\cos\theta$.

9.46 For two point charges q_1 and q_2, the electric field very far away will look like that of a single point charge $q_1 + q_2$. Go the next step beyond this and show that the electric field at large distances will approach a direction such that it points along a line that passes through the "center of charge" (like the center of mass): $(q_1\vec{r}_1 + q_2\vec{r}_2)/(q_1 + q_2)$. What happens to this calculation if $q_2 = -q_1$? You will find the results of problem 9.31 useful. Sketch various cases of course. At a certain point in the calculation, you will probably want to pick a particular coordinate system and to place the charges conveniently, probably one at the origin and the other on the z-axis. You should keep terms in the expansion for the potential up through $1/r^2$ and then take $-\nabla V$. Unless of course you find a better way.
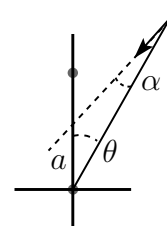
9.47 Fill in the missing steps in deriving Eq. (9.58).

9.48 Analyze the behavior of Eq. (9.58). The first thing you will have to do is to derive the behavior of $\sinh^{-1}$ in various domains and maybe to do some power series expansions. In every case seek an explanation of why the result comes out as it does.
(**a**) If $z = 0$ and $r = \sqrt{x^2 + y^2} \gg L$ what is it and what should it be? (And no, zero won't do.)
(**b**) If $z = 0$ and $r \ll L$ what is it and what should it be?
(**c**) If $z > L$ and $r \to 0$ what is this and what should it be? Be careful with your square roots here.
(**d**) What is the result of (**c**) for $z \gg L$ and for $z - L \ll L$?

9.49 Use the magnetic field equations as in problem 9.22, and assume a current density that is purely azimuthal and dependent on r alone. That is, in cylindrical coordinates it is

$$\vec{J} = J_0 \, \hat{\phi} f(r)$$

Look for a solution for the magnetic field in the form $\vec{B} = \hat{z}\,B_z(r)$. What is the total current per length in this case? That is, for a length Δz how much current is going around the axis and how is this related to B_z along $r = 0$?

Examine also the special case for which $f(r) = 0$ except in a narrow range $a < r < b$ with $b - a \ll b$ (thin). Compare this result to what you find in introductory texts about the solenoid.

9.50 For a spherical mass distribution such as the Earth, what would the mass density function have to be so that the gravitational field has constant strength as a function of depth? Ans: $\rho \propto 1/r$.

9.51 Use index notation to derive

$$(\vec{A} \times \vec{B}) \cdot (\vec{C} \times \vec{D}) = (\vec{A} \cdot \vec{C})(\vec{B} \cdot \vec{D}) - (\vec{A} \cdot \vec{D})(\vec{B} \cdot \vec{C})$$

9.52 Show that $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot \nabla \times \vec{A} - \vec{A} \cdot \nabla \times \vec{B}$. Use index notation to derive this.

9.53 Use index notation to compute $\nabla e^{i\vec{k} \cdot \vec{r}}$. Also compute the Laplacian of the same exponential, $\nabla^2 = \mathrm{div}\ \mathrm{grad}$.

9.54 Derive the force of one charged ring on another, as shown in equation (2.35).

9.55 A point mass m is placed at a distance $d > R$ from the center of a spherical shell of radius R and mass M. Starting from Newton's gravitational law for point masses, derive the force on m from M. Place m on the z-axis and use spherical coordinates to express the piece of dM within $d\theta$ and $d\phi$. (This problem slowed Newton down when he first tried to solve it, as he had to stop and invent integral calculus first.)

9.56 The volume charge density is measured near the surface of a (not quite perfect) conductor to be $\rho(x) = \rho_0 e^{x/a}$ for $x < 0$. The conductor occupies the region $x < 0$, so the electric field in the conductor is zero once you're past the thin surface charge density. Find the electric field everywhere and graph it. Assume that everything is independent of y and z (and t).