UMTG–214

A Spin Chain Primer

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Abstract

This is a very elementary introduction to the Heisenberg (XXX) quantum spin chain, the Yang-Baxter equation, and the algebraic Bethe Ansatz.

1 Introduction

Like the quantum harmonic oscillator, the Heisenberg quantum spin chain is one of the fundamental models of physics. Indeed, it is simply formulated, it describes real systems which are experimentally accessible [1], it has a rich and elegant mathematical structure, it is the prototype of all integrable models, and it has close connection to integrable and conformal field theory. Here we review the formulation of the model, and we describe the elements of the so-called algebraic Bethe Ansatz approach for diagonalizing the Hamiltonian. The aim of these notes is to enable beginning graduate students to start exploring the vast literature on this subject.

2 Heisenberg spin chain

We formulate the spin 1/2 isotropic Heisenberg (XXX) quantum spin chain first for one site, then for two sites, and then finally for N sites.

2.1 One site

Actually, the one-site problem is too trivial: one cannot write down the Heisenberg spin Hamiltonian for only one site. Nevertheless, it is important to identify the set of observables and the vector space on which these operators act.

For the one-site problem, a basis for the observables consists of the identity matrix $\mathbb{I}$ and the familiar Pauli spin matrices $\vec{\sigma}$, that is

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1)$$

These operators act on a two-dimensional complex vector space $V = \mathbb{C}^2$, whose elements we represent by two-component vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, with $x_i \in \mathbb{C}$.

2.2 Two sites

We generalize to the case of more than one site using the notion of tensor product. For the 2×2 matrices $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, the tensor product $A \otimes B$ is defined

[1] See, e.g., F. Essler's contribution to these Proceedings.

by [2]

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \otimes \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \left(\begin{array}{cc|cc} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ \hline a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{array}\right) . \tag{2}$$

For the two-site problem, the basic observables are the spin operators at each site, $\vec{\sigma}_1 \equiv \vec{\sigma} \otimes \mathbb{I}$ and $\vec{\sigma}_2 \equiv \mathbb{I} \otimes \vec{\sigma}$. That is, [3]

$$\sigma_1^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc} & & 1 & \\ & & & 1 \\ \hline 1 & & & \\ & 1 & & \end{array}\right) ,$$

$$\sigma_1^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc} & & -i & \\ & & & -i \\ \hline i & & & \\ & i & & \end{array}\right) ,$$

$$\sigma_1^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right) , \tag{3}$$

and

$$\sigma_2^x = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} & 1 & & \\ 1 & & & \\ \hline & & & 1 \\ & & 1 & \end{array}\right) ,$$

$$\sigma_2^y = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \left(\begin{array}{cc|cc} & -i & & \\ i & & & \\ \hline & & & -i \\ & & i & \end{array}\right) ,$$

[2] I like to draw the horizontal and vertical lines to help organize the matrix elements; they have no other significance, and can be omitted.

[3] For visual clarity, I write only the nonzero matrix elements in the tensor product.

$$\sigma_2^z \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \left(\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & 1 & \\ & & & -1 \end{array}\right) . \tag{4}$$

Note that

$$\left[\sigma_1^i \,, \sigma_2^j\right] = 0 \tag{5}$$

for any $i \,, j \in \{x \,, y \,, z\}$.

These operators act on the tensor product space $V \otimes V$, with elements

$$x \otimes y = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \otimes \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \left(\begin{array}{c} x_1 y_1 \\ x_1 y_2 \\ \hline x_2 y_1 \\ x_2 y_2 \end{array}\right) . \tag{6}$$

This is a good place to introduce the very important permutation matrix $\mathcal{P}$, which is defined by

$$\mathcal{P} = \left(\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right) . \tag{7}$$

In view of Eq. (6),

$$\mathcal{P}\,(x \otimes y) = \left(\begin{array}{c} x_1 y_1 \\ x_2 y_1 \\ \hline x_1 y_2 \\ x_2 y_2 \end{array}\right) = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = y \otimes x \,, \tag{8}$$

and so indeed $\mathcal{P}$ permutes the factors in the tensor product. Evidently, $\mathcal{P}^2 = \mathbb{I} \otimes \mathbb{I}$. Note also that

$$\mathcal{P}\;\vec{\sigma}_1\;\mathcal{P} = \vec{\sigma}_2 \,, \qquad \mathcal{P}\;\vec{\sigma}_2\;\mathcal{P} = \vec{\sigma}_1 \,. \tag{9}$$

The two-site Heisenberg spin Hamiltonian is [4]

$$H_{12} \;=\; \frac{J}{4}\,(\vec{\sigma}_1 \cdot \vec{\sigma}_2 - \mathbb{I} \otimes \mathbb{I}) = \frac{J}{4}\,(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_1^z \sigma_2^z - \mathbb{I} \otimes \mathbb{I})$$

[4] We subtract a constant (proportional to the identity matrix) for later convenience.

$$
\begin{aligned}
&= \frac{J}{4}\left(\sigma^x\otimes\sigma^x+\sigma^y\otimes\sigma^y+\sigma^z\otimes\sigma^z-\mathbb{I}\otimes\mathbb{I}\right)\\
&= \frac{J}{2}\left(\begin{array}{cc|cc} 0&0&0&0\\ 0&-1&1&0\\ \hline 0&1&-1&0\\ 0&0&0&0 \end{array}\right). \qquad (10)
\end{aligned}
$$

We observe that

$$H_{12}=\frac{J}{2}\left(\mathcal{P}-\mathbb{I}\otimes\mathbb{I}\right). \qquad (11)$$

That is, apart from multiplicative and additive constants, the two-site Hamiltonian is given by the permutation matrix.

We consider now the problem of diagonalizing the Hamiltonian,

$$H_{12}|\psi\rangle = E|\psi\rangle\,. \qquad (12)$$

One can verify by inspection that the solution is given by

$$
\begin{aligned}
|\psi_{(1\,,1)}\rangle &= \left(\begin{array}{c} 1\\ 0\\ \hline 0\\ 0 \end{array}\right) = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix},\\
|\psi_{(1\,,0)}\rangle &= \left(\begin{array}{c} 0\\ 1\\ \hline 1\\ 0 \end{array}\right) = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix},\\
|\psi_{(1\,,-1)}\rangle &= \left(\begin{array}{c} 0\\ 0\\ \hline 0\\ 1 \end{array}\right) = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix},\\
|\psi_{(0\,,0)}\rangle &= \left(\begin{array}{c} 0\\ 1\\ \hline -1\\ 0 \end{array}\right) = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}. \qquad (13)
\end{aligned}
$$

The first three states have the same energy $E=0$, and the last state has energy $E=-J$.

This result can be easily understood from the $su(2)$ symmetry of the model. Indeed, the total spin operators

$$\vec{S} \equiv \frac{1}{2}\left(\vec{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \vec{\sigma}\right) \tag{14}$$

are generators of a *reducible* 4-dimensional representation of $su(2)$. That is, there exists a unitary matrix U such that

$$U\ \vec{S}\ U^{\dagger} = \left(\begin{array}{c|c} \vec{S}_{(S=1)} & \\ \hline & \vec{S}_{(S=0)} \end{array}\right), \tag{15}$$

where $\vec{S}_{(S=1)}$ and $\vec{S}_{(S=0)}$ generate irreducible representations of dimension 3 and 1, respectively. Moreover,

$$\begin{aligned} \vec{S}^2 &= \frac{1}{4}\left(\vec{\sigma} \otimes \mathbb{I} + \mathbb{I} \otimes \vec{\sigma}\right)^2 \\ &= \frac{1}{4}\left(\vec{\sigma}^2 \otimes \mathbb{I} + 2\vec{\sigma} \otimes \vec{\sigma} + \mathbb{I} \otimes \vec{\sigma}^2\right) \\ &= \frac{1}{2}\vec{\sigma} \otimes \vec{\sigma} + \frac{3}{2}\mathbb{I} \otimes \mathbb{I}, \end{aligned} \tag{16}$$

since $\vec{\sigma}^2 = 3\mathbb{I}$. Hence the two-site Hamiltonian can be expressed in terms of $\vec{S}^2$,

$$H_{12} = \frac{J}{4}\left(\vec{\sigma} \otimes \vec{\sigma} - \mathbb{I} \otimes \mathbb{I}\right) = \frac{J}{2}\left(\vec{S}^2 - 2\mathbb{I} \otimes \mathbb{I}\right). \tag{17}$$

Recalling the well-known fact $\vec{S}^2|S\,,S^z\rangle = S(S+1)|S\,,S^z\rangle$, we see that

$$H_{12}|S\,,S^z\rangle = \frac{J}{2}\left(S(S+1) - 2\right)|S\,,S^z\rangle\,, \quad S^z = -S\,,\ldots\,,S\,; \quad S = 0\,,1\,. \tag{18}$$

In particular, for $S = 1$ the energy is $E = 0$, and for $S = 0$ the energy is $E = -J$, which is the result we found earlier.

Note that for $J > 0$ (antiferromagnetic), the ground state is a spin singlet ($S = 0$) state; while for $J < 0$ (ferromagnetic), there is a degenerate ground state.

2.3 N sites

For the N-site problem, the basic observables are $\vec{\sigma}_n\,, \quad n = 1\,,2\,,\ldots\,,N$, defined by

$$\vec{\sigma}_n = \overset{1}{\underset{\downarrow}{\mathbb{I}}} \otimes \cdots \otimes \mathbb{I} \otimes \overset{n}{\underset{\downarrow}{\vec{\sigma}}} \otimes \mathbb{I} \otimes \cdots \otimes \overset{N}{\underset{\downarrow}{\mathbb{I}}}\,. \tag{19}$$

These are operators on

$$\overset{1}{\underset{\downarrow}{V}} \otimes \cdots \otimes \overset{n}{\underset{\downarrow}{V}} \otimes \cdots \otimes \overset{N}{\underset{\downarrow}{V}} \tag{20}$$

which act nontrivially on the n^{th} space, and trivially on the rest.

There are two possible topologies for a one-dimensional chain: open or closed. Correspondingly, there are two Heisenberg spin Hamiltonians describing nearest-neighbor interactions:

$$H = \sum_{n=1}^{N-1} H_{n\,,n+1} \qquad \text{(open)} \tag{21}$$

or

$$H = \sum_{n=1}^{N-1} H_{n\,,n+1} + H_{N\,,1} \qquad \text{(closed)} \tag{22}$$

where the two-site Hamiltonian is

$$H_{ij} = \frac{J}{4}\left(\vec{\sigma}_i \cdot \vec{\sigma}_j - \mathbb{I}^{\otimes N}\right) . \tag{23}$$

For simplicity, we shall henceforth focus on the closed chain.

How can one diagonalize the Hamiltonian? Using a computer is not practical for large values of N. Indeed, H is a $2^N \times 2^N$ matrix. Moreover, direct numerical diagonalization will give all eigenstates, while often one is interested in only low-lying states.

A very elegant alternative approach which overcomes (at least in part) both of these difficulties is the Bethe Ansatz, of which there are several variants. We focus here on the algebraic Bethe Ansatz. [5] An essential element of this approach is a matrix $R(\lambda)$ that is a solution of the Yang-Baxter equation, which we discuss in Section 3. As we shall see in Section 4, the two-site Hamiltonian (23) can be expressed in terms of the R matrix. The fact that the R matrix satisfies the Yang-Baxter equation leads to the "integrability"of the model, i.e., that the model can be solved by Bethe Ansatz.

3 Yang-Baxter equation

We consider the R matrix [6]

$$R(\lambda) \;=\; \lambda\mathbb{I}\otimes\mathbb{I} + i\mathcal{P}$$

[5] The method pioneered by Bethe [1] is now known as coordinate Bethe Ansatz. The algebraic Bethe Ansatz was developed in St. Petersburg [2] - [6]. It is perhaps not as powerful as coordinate Bethe Ansatz, but it is more transparent. Other related approaches include analytic Bethe Ansatz [7] and Baxter's Q-operator method [8]. A rather different approach, which is formulated directly in the thermodynamic limit and avoids Bethe Ansatz, is being developed in Kyoto [9].

[6] The notation [4] which we are using, while widely used, has not been universally adopted. In particular, some authors (e.g., [2], [5], [6]) call $R(\lambda)$ what we call $\check{R}(\lambda) \equiv \mathcal{P}R(\lambda)$. Given an R matrix, one can easily determine which of the two conventions is being used by evaluating it at $\lambda = 0$: for us $R(0) \sim \mathcal{P}$, while for them $R(0) \sim \mathbb{I}\otimes\mathbb{I}$. Of course, their Yang-Baxter equation looks somewhat different from ours.

$$= \left(\begin{array}{cc|cc} \lambda+i & & & \\ & \lambda & i & \\ \hline & i & \lambda & \\ & & & \lambda+i \end{array}\right) = \left(\begin{array}{cc|cc} a & & & \\ & b & c & \\ \hline & c & b & \\ & & & a \end{array}\right), \tag{24}$$

where

$$a = \lambda + i\,, \qquad b = \lambda\,, \qquad c = i\,. \tag{25}$$

We regard $R(\lambda)$ as an operator which acts on $V \otimes V$. The variable λ is called the spectral parameter.

We wish to show that this matrix is a solution to the so-called Yang-Baxter equation. [7] To this end, we define $R_{12}(\lambda)$ by

$$R_{12}(\lambda) \;=\; R(\lambda)\otimes\mathbb{I} = \left(\begin{array}{cc|cc} a & & & \\ & b & c & \\ \hline & c & b & \\ & & & a \end{array}\right) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \left(\begin{array}{cc|cc|cc} a & & & & & \\ & a & & & & \\ \hline & & b & c & & \\ & & & b & c & \\ \hline & & c & & b & \\ & & & c & & b \\ \hline & & & & & a \\ & & & & & & a \end{array}\right). \tag{26}$$

Evidently, this is an operator on $V \otimes V \otimes V$, which acts nontrivially on the first and second spaces, and trivially on the third.

Similarly, we define $R_{23}(\lambda)$ by

$$R_{23}(\lambda) \;=\; \mathbb{I}\otimes R(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \left(\begin{array}{cc|cc} a & & & \\ & b & c & \\ \hline & c & b & \\ & & & a \end{array}\right)$$

[7] For the case of the simple R matrix (24), the approach we shall follow is unnecessarily tedious – there are much faster ways to show that this matrix is a solution of the Yang-Baxter Eq. (31). However, our approach has the benefit of being explicit and straightforward, and it works for any case.

$$= \left(\begin{array}{c|c|c|c} a & & & \\ & b\ c & & \\ \hline & c\ b & & \\ & & a & \\ \hline & & a & \\ & & & b\ c \\ \hline & & & c\ b \\ & & & & a \end{array}\right) . \tag{27}$$

This operator acts nontrivially on the second and third spaces, and trivially on the first.

Finally, we wish to define $R_{13}(\lambda)$, which acts nontrivially on the first and third spaces, and trivially on the second. This requires only slightly more effort:

$$R_{13}(\lambda) = \mathcal{P}_{23}\ R_{12}(\lambda)\ \mathcal{P}_{23}\,, \tag{28}$$

where $R_{12}(\lambda)$ is given by Eq. (26), and $\mathcal{P}_{23}$ is given by

$$\mathcal{P}_{23} \;=\; \mathbb{I} \otimes \mathcal{P} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \left(\begin{array}{c|c} 1 & \\ \hline & 1 \\ \hline 1 & \\ \hline & 1 \end{array}\right)$$

$$= \left(\begin{array}{c|c|c|c} 1 & & & \\ & 1 & & \\ \hline & & & \\ & & & \\ \hline & & 1 & \\ & & & 1 \\ \hline & & & 1 \\ & & & & 1 \end{array}\right) . \tag{29}$$

One can now easily obtain

$$R_{13}(\lambda) = \left(\begin{array}{c|c|c|c} a & & & \\ & b & & c \\ \hline & & a & \\ & & & b \\ \hline & c & & b \\ & & & & a \\ \hline & & c & \\ & & & & b \\ & & & & & a \end{array}\right) . \tag{30}$$

The Yang-Baxter equation is

$$R_{12}(\lambda-\lambda')\ R_{13}(\lambda)\ R_{23}(\lambda') = R_{23}(\lambda')\ R_{13}(\lambda)\ R_{12}(\lambda-\lambda')\,. \tag{31}$$

It is now a tedious but straightforward exercise in matrix multiplication and algebra to show that this relation is satisfied. With the help of a symbolic manipulation program such as Mathematica, the task can be accomplished in minutes. [8]

4 Transfer matrix

In this Section, we show that the two-site Heisenberg Hamiltonian can be expressed in terms of the R matrix, and that the model is integrable by virtue of the fact that this matrix is a solution of the Yang-Baxter equation. However, it is more convenient to work "backwards": using the R matrix, we first construct the so-called transfer matrix – a one-parameter commutative family of operators acting on the space of states of the Heisenberg spin chain. We then verify that the Heisenberg Hamiltonian is among this family of commuting operators.

The key step in this program is to introduce the so-called L operators [9]

$$\begin{aligned} L_{0n}(\lambda) &= R_{0n}(\lambda-\frac{i}{2}) \\ &= \begin{pmatrix} \alpha_n & \beta_n \\ \gamma_n & \delta_n \end{pmatrix}, \qquad n=1\,,2\,,\ldots\,,N\,, \end{aligned} \tag{32}$$

which act on so-called auxiliary (0) and quantum (n) spaces. That is, L_{0n} is an operator on

$$\overset{0}{\downarrow}V \otimes \overset{1}{\downarrow}V \otimes \cdots \otimes \overset{n}{\downarrow}V \otimes \cdots \otimes \overset{N}{\downarrow}V \tag{33}$$

which acts nontrivially on the 0^{th} and n^{th} spaces, and trivially on the rest. Moreover,

$$\begin{aligned} \alpha_n &= \overset{1}{\downarrow}\mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \overset{n}{\downarrow}\alpha \otimes \mathbb{I} \otimes \cdots \otimes \overset{N}{\downarrow}\mathbb{I}\,, \\ \beta_n &= \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \beta \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}\,, \\ \gamma_n &= \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \gamma \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}\,, \\ \delta_n &= \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \delta \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}\,, \end{aligned} \tag{34}$$

[8] The R matrix (24) is the simplest solution of the Yang-Baxter equation. Many more solutions are known. See, e.g., [4], [10].

[9] The shift in λ is made in order that the Bethe Ansatz equations (see Eq. (61) below) have a symmetric form. This shift is not essential.

where

$$\begin{aligned}
\alpha &= \begin{pmatrix} \lambda+\frac{i}{2} & 0 \\ 0 & \lambda-\frac{i}{2} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \\
\gamma &= \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \qquad \delta = \begin{pmatrix} \lambda-\frac{i}{2} & 0 \\ 0 & \lambda+\frac{i}{2} \end{pmatrix}.
\end{aligned} \tag{35}$$

Evidently, α_n, β_n, γ_n, δ_n are operators on

$$\overset{1}{\underset{\downarrow}{V}} \otimes \cdots \otimes \overset{n}{\underset{\downarrow}{V}} \otimes \cdots \otimes \overset{N}{\underset{\downarrow}{V}} \tag{36}$$

which act nontrivially on the n^{th} space, and trivially on the rest. The vector space (36) is precisely the space of states of the Heisenberg quantum spin chain (20). Note that the operators at site n commute with the operators at any other site $n' \neq n$.

The algebra of the operators α_n, β_n, γ_n, δ_n is encoded in the relation

$$R_{00'}(\lambda-\lambda')\; L_{0n}(\lambda)\; L_{0'n}(\lambda') = L_{0'n}(\lambda')\; L_{0n}(\lambda)\; R_{00'}(\lambda-\lambda')\,, \tag{37}$$

which immediately follows from the Yang-Baxter Eq. (31) upon replacing $1 \to 0$, $2 \to 0'$, and $3 \to n$, and also making the shifts $\lambda \to \lambda - \frac{i}{2}$ and $\lambda' \to \lambda' - \frac{i}{2}$.

The monodromy matrix $T_0(\lambda)$ is defined as the following product of L operators [10]

$$\begin{aligned}
T_0(\lambda) &= L_{0N}(\lambda) \cdots L_{01}(\lambda) \\
&= \begin{pmatrix} \alpha_N & \beta_N \\ \gamma_N & \delta_N \end{pmatrix} \cdots \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}.
\end{aligned} \tag{38}$$

The monodromy matrix obeys the fundamental relation

$$R_{00'}(\lambda-\lambda')\; T_0(\lambda)\; T_{0'}(\lambda') = T_{0'}(\lambda')\; T_0(\lambda)\; R_{00'}(\lambda-\lambda')\,. \tag{39}$$

We prove this result for $N = 2$. The left hand side (LHS) is then

$$\begin{aligned}
LHS &= R_{00'}(\lambda-\lambda')\; L_{02}(\lambda)\; L_{01}(\lambda)\; L_{0'2}(\lambda')\; L_{0'1}(\lambda') \\
&= R_{00'}(\lambda-\lambda')\; L_{02}(\lambda)\; L_{0'2}(\lambda')\; L_{01}(\lambda)\; L_{0'1}(\lambda') \\
&= L_{0'2}(\lambda')\; L_{02}(\lambda)\; R_{00'}(\lambda-\lambda')\; L_{01}(\lambda)\; L_{0'1}(\lambda') \\
&= L_{0'2}(\lambda')\; L_{02}(\lambda)\; L_{0'1}(\lambda')\; L_{01}(\lambda)\; R_{00'}(\lambda-\lambda') \\
&= L_{0'2}(\lambda')\; L_{0'1}(\lambda')\; L_{02}(\lambda)\; L_{01}(\lambda)\; R_{00'}(\lambda-\lambda') = RHS\,.
\end{aligned} \tag{40}$$

[10] As is customary, we often suppress the quantum-space subscripts.

In passing to the second line, we have used the fact that L_{0n} commutes with $L_{0'n'}$ for $n \neq n'$, and in passing to the third line we have used the fact (37). This proof immediately extends to the case of general N.

The transfer matrix $t(\lambda)$ is defined by tracing over the auxiliary space

$$t(\lambda) = \mathrm{tr}_0\, T_0(\lambda)\,. \tag{41}$$

It acts only on the quantum spaces (36). The transfer matrix constitutes a one-parameter family of commuting operators

$$[t(\lambda)\,, t(\lambda')] = 0\,. \tag{42}$$

Indeed, multiplying the fundamental relation (39) on the right by the inverse R matrix, and taking the trace over both auxiliary spaces 0 and $0'$, we obtain

$$\mathrm{tr}_{00'}\, R_{00'}(\lambda - \lambda')\; T_0(\lambda)\; T_{0'}(\lambda')\; R_{00'}(\lambda - \lambda')^{-1} = \mathrm{tr}_{00'}\, T_{0'}(\lambda')\; T_0(\lambda)\;\; . \tag{43}$$

Using the cyclic property of the trace, we obtain

$$\mathrm{tr}_{00'}\, T_0(\lambda)\; T_{0'}(\lambda') = \mathrm{tr}_{00'}\, T_{0'}(\lambda')\; T_0(\lambda)\,, \tag{44}$$

which implies the result (42).

The transfer matrix contains the momentum operator P of the (closed) quantum spin chain. Specifically,

$$P = \frac{1}{i} \log\; i^{-N} t(\frac{i}{2})\,. \tag{45}$$

We show this for the case $N = 2$. We first observe

$$t(\frac{i}{2}) = \mathrm{tr}_0\, T_0(\frac{i}{2}) = \mathrm{tr}_0\, R_{02}(0)\; R_{01}(0) = i^2\, \mathrm{tr}_0\, \mathcal{P}_{02}\; \mathcal{P}_{01} = -\mathcal{P}_{12}\,, \tag{46}$$

where we have used the fact $R(0) = i\mathcal{P}$ (see Eq. (24)) and the identity $\mathrm{tr}_0\, A_0 \mathcal{P}_{0n} = A_n$. It follows that

$$t(\frac{i}{2})\; X_1\; t(\frac{i}{2})^{-1} = \mathcal{P}_{12}\; X_1\; \mathcal{P}_{12} = X_2\,, \tag{47}$$

where X_n is any operator at site n. One can show for general N in similar fashion that $t(\frac{i}{2})$ is the one-site shift operator,

$$t(\frac{i}{2})\; X_n\; t(\frac{i}{2})^{-1} = X_{n+1}\,, \qquad n = 1\,, \ldots\,, N\,. \tag{48}$$

We recall from elementary quantum mechanics that

$$e^{iaP}\; X\; e^{-iaP} = X + a\,, \tag{49}$$

where P and X are the momentum and position operators, and a is a c-number. By analogy, we define the momentum operator for a spin chain by $e^{iP} = i^{-N} t(\frac{i}{2})$, which implies (45).

The transfer matrix also contains a closed-chain Hamiltonian

$$H = \frac{J}{2}\left(i\frac{d}{d\lambda}\log t(\lambda)\Big|_{\lambda=\frac{i}{2}} - N\mathbb{I}^{\otimes N}\right) = \sum_{n=1}^{N-1} H_{n\,,n+1} + H_{N\,,1}\,, \tag{50}$$

with the two-site Hamiltonian

$$H_{ij} = \frac{J}{2}\left(\mathcal{P}_{ij}\ R'_{ij}(0) - \mathbb{I}^{\otimes N}\right)\,, \tag{51}$$

where the prime denotes differentiation with respect to λ. This is easily verified for $N = 2$ using (46) and also

$$t'(\frac{i}{2}) = \mathrm{tr}_0\, R'_{02}(0)\ R_{01}(0) + \mathrm{tr}_0\, R_{02}(0)\ R'_{01}(0) = 2iR'_{12}(0)\,. \tag{52}$$

Moreover, keeping in mind the result (11) and the expression (24) for our R matrix, we see that the two-site Hamiltonian (51) coincides with that of the Heisenberg chain (23). In short, the Hamiltonian (50) is precisely the Heisenberg Hamiltonian (22).

Combining the results (42) and (50), we see that the Hamiltonian commutes with the transfer matrix,

$$[H\,, t(\lambda)] = 0\,. \tag{53}$$

This relation together with (42) signal the "integrability" of the model.

5 Algebraic Bethe Ansatz

We finally turn to the problem of diagonalizing the Hamiltonian. In fact, we shall diagonalize the transfer matrix. The strategy is to identify certain creation operators with which to construct states, as one does for the simple harmonic oscillator. These operators, however, depend on parameters; the states are eigenstates of the transfer matrix if the parameters are solutions of a set of equations, namely the Bethe Ansatz equations.

The monodromy matrix $T_0(\lambda)$ is a 2×2 matrix in the auxiliary space, whose elements are operators on the quantum space $V^{\otimes N}$ which we denote as follows

$$T_0(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}\,. \tag{54}$$

A set of algebraic relations among these four operators is encoded in the fundamental relation (39). The relations include

$$[B(\lambda)\,,B(\lambda')]=0\,,$$

$$\begin{aligned} A(\lambda)\;B(\lambda') &= \frac{a(\lambda'-\lambda)}{b(\lambda'-\lambda)}B(\lambda')\;A(\lambda)-\frac{c(\lambda'-\lambda)}{b(\lambda'-\lambda)}B(\lambda)\;A(\lambda')\,,\\ D(\lambda)\;B(\lambda') &= \frac{a(\lambda-\lambda')}{b(\lambda-\lambda')}B(\lambda')\;D(\lambda)-\frac{c(\lambda-\lambda')}{b(\lambda-\lambda')}B(\lambda)\;D(\lambda')\,, \end{aligned} \tag{55}$$

where $a\,,b\,,c$ are given by Eq. (25).

Let ω_+ be the ferromagnetic vacuum state with all spins up,

$$\omega_+=\underbrace{\begin{pmatrix}1\\0\end{pmatrix}\otimes\cdots\otimes\begin{pmatrix}1\\0\end{pmatrix}}_{N}\,, \tag{56}$$

which is an eigenstate of $A(\lambda)$ and $D(\lambda)$, and which is annihilated by $C(\lambda)$,

$$A(\lambda)\;\omega_+=\left(\lambda+\frac{i}{2}\right)^N\omega_+\,,\qquad D(\lambda)\;\omega_+=\left(\lambda-\frac{i}{2}\right)^N\omega_+\,,\qquad C(\lambda)\;\omega_+=0\,. \tag{57}$$

We use the operators $B(\lambda)$ as creation operators to construct the so-called Bethe state

$$|\lambda_1\,,\ldots\,,\lambda_M\rangle=B(\lambda_1)\cdots B(\lambda_M)\;\omega_+\,. \tag{58}$$

Using the algebraic relations (55) and the properties (57) of the ferromagnetic vacuum state, it can be shown that the Bethe state is an eigenstate of the transfer matrix $t(\lambda)=A(\lambda)+D(\lambda)$,

$$t(\lambda)\;|\lambda_1\,,\ldots\,,\lambda_M\rangle=\Lambda(\lambda\,;\lambda_1\,,\ldots\,,\lambda_M)\;|\lambda_1\,,\ldots\,,\lambda_M\rangle \tag{59}$$

with the eigenvalue

$$\Lambda(\lambda\,;\lambda_1\,,\ldots\,,\lambda_M)=\left(\lambda+\frac{i}{2}\right)^N\prod_{\alpha=1}^{M}\frac{\lambda-\lambda_\alpha-i}{\lambda-\lambda_\alpha}+\left(\lambda-\frac{i}{2}\right)^N\prod_{\alpha=1}^{M}\frac{\lambda-\lambda_\alpha+i}{\lambda-\lambda_\alpha}\,, \tag{60}$$

if $\{\lambda_1\,,\ldots\,,\lambda_M\}$ are distinct and obey the Bethe Ansatz equations

$$\left(\frac{\lambda_\alpha+\frac{i}{2}}{\lambda_\alpha-\frac{i}{2}}\right)^N=\prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{M}\frac{\lambda_\alpha-\lambda_\beta+i}{\lambda_\alpha-\lambda_\beta-i}\,,\qquad \alpha=1\,,\cdots\,,M\,. \tag{61}$$

The proof is immediate for $N=1$, and is not much more difficult for general N. It is well-described in many papers (see, e.g., [2],[3],[6]), so we shall not repeat it here.

In particular, one can now obtain the momentum and energy eigenvalues using Eqs. (45), (50), and (60),

$$P = \frac{1}{i}\sum_{\alpha=1}^{M} \log\left(\frac{\lambda_\alpha + \frac{i}{2}}{\lambda_\alpha - \frac{i}{2}}\right) \pmod{2\pi}\,, \qquad E = -\frac{J}{2}\sum_{\alpha=1}^{M} \frac{1}{\lambda_\alpha^2 + \frac{1}{4}}\,. \tag{62}$$

An important feature of the Heisenberg spin chain is its $su(2)$ symmetry. One can show [5] that the transfer matrix commutes with the total spin operators,

$$\left[\vec{S}\,, t(\lambda)\right] = 0\,, \qquad \vec{S} = \frac{1}{2}\sum_{n=1}^{N} \vec{\sigma}_n\,. \tag{63}$$

Moreover, the Bethe states (58) are $su(2)$ highest weight states

$$S^+ \,|\lambda_1\,, \ldots\,, \lambda_M\rangle = 0\,, \qquad S^\pm = S^x \pm i S^y\,, \tag{64}$$

and are eigenstates of S^z with eigenvalue

$$S^z = \frac{N}{2} - M\,. \tag{65}$$

Since $S = S^z \geq 0$, it follows that $M \leq N/2$. Lower weight states are obtained by acting on the Bethe states with the lowering operator S^-.

6 Now the hard work begins

The $S = 1/2$ isotropic Heisenberg spin chain which we have discussed is the simplest integrable magnetic chain. Many generalizations are possible. For instance, using appropriate R matrices, one can construct anisotropic (XXZ and XYZ) chains, chains with spins in higher-dimensional representations ($S = 1\,, 3/2\,, \ldots$) or with combinations of different spins, and chains with spins in representations of higher-rank algebras ($su(\mathcal{N})\,, \ldots$). Using also so-called K matrices, one can construct integrable open spin chains [11]. Bethe Ansatz equations have been obtained for many such models. [11]

Obtaining Bethe Ansatz equations (BAE) for an integrable model is in a sense "half" the solution of the model – the easier half at that! The difficulty is that quantities of physical interest (e.g., energy) are expressed in terms of solutions of BAE; and unfortunately, BAE are generally hard to solve. Various clever methods have been developed for extracting from these equations information about the thermodynamic ($N \to \infty$) limit. Invariably the methods involve some assumptions about the nature of the solutions, such as the string

[11] Numerous additional references can be found in [6].

hypothesis (see, e.g., [5]). The types of results which have been obtained include spectrum of low-lying states, scattering matrices, finite-size corrections, and thermodynamics (finite temperature and magnetic field). Progress has been made on the computation of correlation functions. See, e.g., [6] and other contributions to these Proceedings.

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